The Winnability of Klondike Solitaire and Many Other Patience Games

Charlie Blake thecharlieblake@gmail.com
Work undertaken while at School of Computer Science, University of St Andrews, St Andrews, UK
Ian P. Gent ian.gent@st-andrews.ac.uk
School of Computer Science, University of St Andrews, St Andrews, UK (Corresponding Author)

Abstract

Our ignorance of the winnability percentage of the solitaire card game 'Klondike' has been described as "one of the embarrassments of applied mathematics" by Yan, Diaconis, Rusmevichientong, and Roy. Klondike, the game in the Windows Solitaire program, is just one of many single-player card games, generically called 'patience' or 'solitaire' games, for which players have long wanted to know how likely a particular game is to be winnable. A number of different games have been studied empirically in the academic literature and by non-academic enthusiasts. Here we show that a single general purpose Artificial Intelligence program named 'Solvitaire' can be used to determine the winnability percentage of 73 variants of 35 different single-player card games with a 95% confidence interval of ± 0.1% or better. For example, we report the winnability of Klondike as 81.945% ± 0.084% (in the 'thoughtful' variant where the player knows the rank and suit of all cards), a 30-fold reduction in confidence interval over the best previous result. The vast majority of our results are either entirely new or represent significant improvements on previous knowledge.

Authors' Note: *An earlier version of this paper was put on arXiv in June 2019. This version is significantly extended with new research and much greater detail given in several areas. All statements are correct as of August 2024, to the best of our belief.*

1. Introduction

Patience games - single-player card games also known as 'solitaire' games[1] - have been a popular pastime for more than 200 years (Ross & Healey, 1963). This popularity continues, with Microsoft Windows Solitaire – just one implementation of one patience game – being played 100 million times per day in 2020 (Jensen, 2020). We compute winnability percentages on random instances of many single-deck patience games using a general solver named 'Solvitaire'. Almost all our results are either entirely new or significant improvements on previous knowledge. Where results were previously known, they were obtained using solvers specific to a particular game or small family of games. In contrast, Solvitaire solves a wide variety of patience games expressible in our flexible rule-description language. Based on depth-first backtracking search, it exploits a number of techniques to improve efficiency: transposition tables (Greenblatt, Eastlake, & Crocker, 1967; Smith, 2005), symmetry (Gent, Petrie, & Puget, 2006), dominances (Chu & Stuckey, 2015), and streamliners (Gomes & Sellmann, 2004; Wetter, Akgün, & Miguel, 2015).

Klondike,[2] the game in Windows Solitaire, is just one example of hundreds of patience games that exist (Parlett, 1980). Understanding the range of games available requires understanding some key terminology: we give a very concise introduction in Section 2.

1. Herein we use the word 'patience' as the traditional word in UK English while 'solitaire' is the US usage (Ross & Healey, 1963).
2. In the main text of this paper, we distinguish names of games by writing them in italics, e.g. *Klondike*.

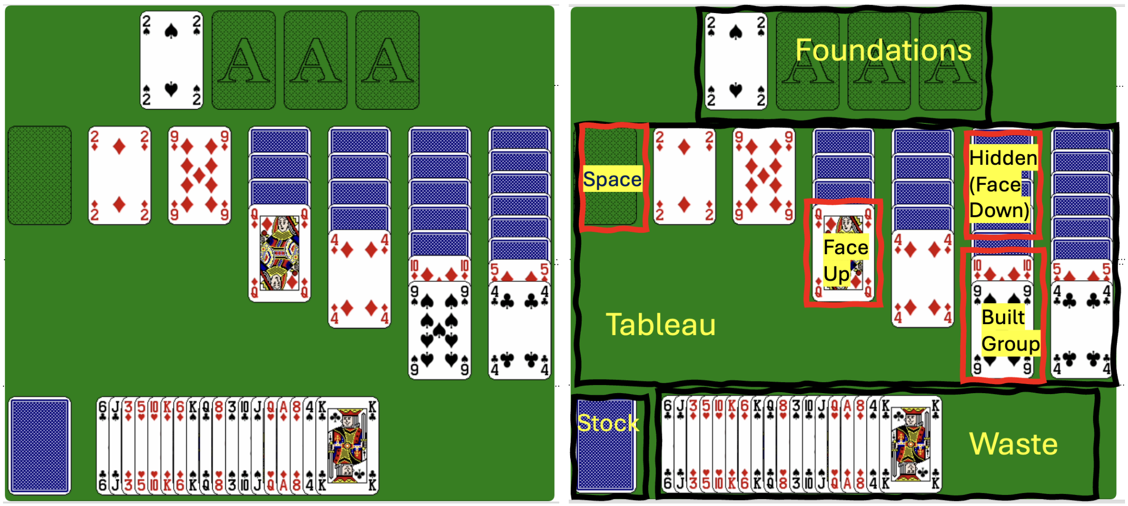

Figure 1: (Left) Sample layout of the game of *Klondike* part way through play. (Right) The same layout illustrating some terminology from Section 2 with general areas of the layout outlined in black, and specific features outlined in red.

The probability of winning has always been of interest to players, with advice published as to how likely a given game is to be winnable at least as long ago as 1890 (Cavendish, 1890). In this paper, we study 81 variants of about 40 different patience games. Not knowing the winnability of just one of these games, *Klondike*, has been called "one of the embarrassments of applied mathematics" (Yan et al., 2005). Only for a very small number of games, e.g. *FreeCell* (Fish, 2018), has this probability previously been known to a high degree of accuracy. For games with hidden cards, we follow standard practice in the literature of considering the 'thoughtful' variant (Yan et al., 2005), in which the ranks and suits of hidden cards are known to the player at the start of the game.

We are now able to report the winnability percentage of thoughtful *Klondike* and dozens of other games with a 95% confidence interval within ±0.1%. Remarkably, we achieve this with a solver which can be used for a very wide variety of games and is not highly optimised for any particular one. Our rule-sets and solver are flexible enough to include famous games such as *Klondike*, *Canfield*, *FreeCell*, *Spider*, *Golf*, *Accordion*, *Black Hole* and *King Albert*, all of which are very different from each other. We are not aware of any previous solver which can be used unchanged on any two of these games.

2. Terminology of Patience Games

Giving a general introduction to single-player card games is outwith the scope of this paper. Excellent introductions to patience games can easily be found in books (Parlett, 1980) or online. A sample game of Klondike together with an illustration of some relevant terminology is shown in Figure 1. Because terminology of patience games is not always the same in different sources, we briefly define key terms we use in this paper.

We use the word **'game'** to refer to a particular set of rules for playing patience. A game is played with a number of complete **'decks'** of cards, normally the standard deck with 13 cards of each of 4 suits. The rules of a game specify how the cards are placed before play starts: in the initial position some cards may be **'hidden'** from the player, for example by being placed **'face-down'**. In this paper we follow previous work in studying

'thoughtful' variants of a game where the ranks and suits of hidden cards are known. With physical cards, the thoughtful variation is like the player peeking at each hidden card to see what it is. Electronic implementations with unlimited undos also become thoughtful, because the player can always go back to the start after finding any information they need in the game. We use the word **'instance'** of a game to refer a particular arrangement of cards for that game, usually after random shuffling. Most games are won by rearranging cards so as to place them in order on a set of **'foundations'**, typically from A to K within each suit: in some cases the player is given some cards already placed on the foundation as a starter. In some games the goal instead is to move cards into a single **'hole'**, with consecutive cards required to be adjacent in rank but with no regard to suit: games vary whether one is allowed to loop round from K to A and vice versa. In some games, like *Spider*, cards are not built to foundations individually but all simultaneously when a complete sequence from A to K in a single suit has been constructed. An instance of a game is **'winnable'** if there is any legal sequence of moves that leads to the goal predetermined by the rules of the game. In most games the main area of play is called the **'tableau'**. Cards can often be moved within the tableau: this is called **'building'** one card onto another pile. In the scope of this paper, the card must be one lower in rank than the card it is placed on (with K considered one lower than A if appropriate). The **'build policy'** determines additional rules: to be built on a card may need to be the same suit as the higher card, or of a suit of the opposite colour, or it may be allowed to be any suit. A sequence of consecutive built cards may be allowed to be moved together as one **'group'**: where allowed this may be with same restriction as the build policy, or sometimes a stricter restriction that the group must all be the same suit. We refer to the number of tableau piles and their sizes at the start of the game as the **'layout'**. Typically a face-down card in the tableau is turned **'face-up'** only when the card immediately covering it is moved. When a tableau pile becomes empty it is called a **'space'**: some games allow cards to be placed in spaces; sometimes the card placed in the space must be a K and in other games any card is allowed. In some games cards may be **'worried back'**: this means cards can be moved from a foundation to the tableau. Some games contain an ordered **'stock'** of cards: often the player is allowed to **'draw'** a given number of cards at a time. Most often stock cards are moved to a **'waste'** pile, from which the top card can then be played to the tableau, while in others one card is dealt onto each tableau pile. Some games allow **'redeals'**, where the waste pile may be reused to form the stock again. Some games have a **'reserve'** of cards which can be played onto the tableau or foundations but otherwise are static. **'Free cells'** function like a reserve but cards may be moved from the tableau into free cells as well as in the other direction.

As an example we can now describe *Klondike* as illustrated in Figure 1. A single standard deck is used and the goal is to build all cards on foundations in suit from A to K. The game begins with a tableau of 28 cards in a triangular form with piles from 1 to 7 cards, with all but the top card face-down. Face-up cards on the tableau may be built in alternating colour, and built groups may be moved. Face-down cards may not be moved.[3] Spaces may be filled only by a K. Cards may be worried back from foundations to tableau. A stock of 24 cards may be drawn in groups of three, and redeals are allowed without limit. This description may be compared with Table 6, Appendix A. These rules are given to Solvitaire in a JSON format shown in Listing 1, page 12: for more on of our rules language, see Section 5.1.1 and Appendix F.

3. This prohibition means that thoughtful Klondike is subtly different from a variant in which all cards start face-up.

Names of particular patience games are even more confused than the terminology for rules, with different names used for the same game and the same name used for different games. For example, the game we call *Klondike* in this paper is often just called 'Patience' (Parlett, 1980) or 'Solitaire' which are also names for the general family of single-player card games. Worse than that, *Klondike* can also be called 'Canfield' which is the name we use here for a completely different game. Both games have many other names, for example both being sometimes called 'Demon'.[4] Unfortunately this means we sometimes do not know what game is being referred to in historical documents: for example Stanislaw Ulam may have been referring to either game when he wrote that 'Canfield Solitaire' motivated his invention of Monte Carlo methods (Eckhardt, 1987). It is therefore particularly important for us to be clear on the name and rules we use for each game. We provide a concise summary of rules we used of most games studied in this paper in Table 6. Almost all games we studied can be described in this way, including all games for which we give the first reported results. The exceptional games that cannot be described in this framework are Accordion (BVS Development Corporation, 2003) and its variant with 18 cards (called 'Late-Binding Solitaire' by its originator) (Ross & Knuth, 1989), the two variants of Gaps (Helmstetter & Cazenave, 2004): their rules can be found in the papers just cited and are shown in our JSON format in Listings 2 and 3, page 2.

3. History of Solving Patience Games

The winnability of patience games has interested people for many years, with many books on the topic providing estimates of how often each game can be won. In some cases, an expert's views were astonishingly accurate: in the nineteenth century Cavendish (1890) said that the game *Fan* "with careful play, is slightly against the player", while we show that it is $48.776\% \pm 0.099\%$ winnable.[5] Other stated claims have been very inaccurate: (British) *Canister* was described by Parlett (1980) as "odds in favour", while Table 2 shows that only slightly more than one in a million games are winnable. Distinguished scientists who have taken an interest in the question include Stanislaw Ulam, the inventor of computer-based Monte Carlo Methods, (Eckhardt, 1987), Donald Knuth, a Turing Award winner (Ross & Knuth, 1989), and Irving Kaplansky, a President of the American Mathematical Society (Mackenzie & Graham, 2019).

There are some patience games where there is no player choice required and the pleasure of the game is the purely mechanical playing out of the game. We do not pay attention to such games in this paper, but some solvability percentages have been calculated. For example, *Clock Patience* is provably won exactly $\frac{1}{13}$ of the time (Jenkyns & Muller, 1981). Monte Carlo methods have shown *Perpetual Motion* to have a winnability of $8.6692 \pm 0.0017\%$ while superficially minor changes to the rules can increase this to $54.8033 \pm 0.0031\%$ winnability (Masten, 2022b; Clarke, 2009).

When Microsoft released one of the early versions of *FreeCell*, it included 32,000 different instances. It was conjectured that all were winnable, leading to an early example of internet crowdsourcing, the 'Internet FreeCell Project' led by Dave Ring in 1994-5 (Plante, 2012).[6] People shared their solutions online for all instances except one, deal number 11982, which nobody could solve. This is now known to be unwinnable (Keller,

4. 'Demon' was the name used by Ian Gent's mother for *Canfield* and by his father for *Klondike*.
5. We studied a very minor variant of Cavendish's game, with sixteen piles of three and two piles of two instead of seventeen piles of three and one of one.
6. Indeed the word 'crowdsourcing' itself was not coined until 10 years later (Howe, 2006).

2015). At a similar time, Don Woods obtained an estimate of 99.999% winnability for *FreeCell* from a computer study of a million random instances (Keller, 2015).[7]

Since then, more computational experiments have given winnability estimates for a variety of games. These have been done both inside and outside the academic community. Some games have attracted academic attention, including *Klondike* (Bjarnason, Fern, & Tadepalli, 2009; Bjarnason, Tadepalli, & Fern, 2007; Yan et al., 2005), *FreeCell* (Paul & Helmert, 2016; Elyasaf, Hauptman, & Sipper, 2012; Dunphy & Heywood, 2003), *Gaps* (Helmstetter & Cazenave, 2004), *King Albert* (Roscoe, 2016) and *Black Hole* (Gent, Jefferson, Kelsey, Lynce, Miguel, Nightingale, Smith, & Tarim, 2007; Smith, 2005). For most patience games, however, the best known winnability estimates have been obtained by enthusiasts rather than academic or industrial researchers. Of games just mentioned, this includes *FreeCell* (Fish, 2018), and included *Klondike* (Birrell, 2017) and *Black Hole* (Fish, 2010) until the current paper. We are not aware of any academic predecessor to our work which studied a diverse range of patience games, but there have been substantial efforts across a range of games by enthusiasts including Shlomi Fish, Mark Masten (2022c), and Jan Wolter (2013b), among others. In summary, the world has owed far more to non-academic than academic research in knowing the winnability of patience games. We have used ideas from both academic and non-academic researchers. For example, for *Klondike* and *Canfield* we made essential use of both the K^+ representation of stock from academic research (Bjarnason et al., 2007) and the dominance described in Section 5.4.2 from non-academic research (Wolter, 2014d; Birrell, 2017). Note, however, that this paper is not intended to give a complete survey of either academic or non-academic work on patience games.

We close this brief history with a remarkable echo in our work of the origin of computer-based Monte Carlo methods, which are the method we use to compute winnability estimates throughout this paper. Monte Carlo methods were actually invented by Stanislaw Ulam with the idea of calculating the winnability of solitaire games, as he recalled:

> *The first thoughts and attempts I made to practice [the Monte Carlo method] were suggested by a question which occurred to me in 1946 as I was convalescing from an illness and playing solitaires. The question was what are the chances that a Canfield laid out with 52 cards will come out successfully? After spending a lot of time trying to estimate them by pure combinatorial calculations, I wondered whether a more practical method than "abstract thinking" might not be to lay it out say one hundred times and simply observe and count the number of successful plays. This was already possible to envisage with the beginning of the new era of fast computers, and I immediately thought of problems of neutron diffusion and other questions of mathematical physics, and more generally how to change processes described by certain differential equations into an equivalent form interpretable as a succession of random operations.*
>
> – Stanislaw Ulam, unpublished remarks 1983, quoted by Eckhardt (1987).

In this paper, we therefore achieve the idea of Ulam, giving a very precise estimate of the winnability of the solitaire he was playing using precisely the Monte Carlo methods he invented to achieve this. As discussed above, we do not know whether Ulam's 'Canfield' was the game we call *Klondike* or *Canfield*. Whichever it may be, in this paper we have reduced the uncertainty of its winnability by a factor of more than 30 over the previous best estimate and to with a 95% confidence interval within ±0.1%.

7. This is a pleasing example of a case where '99.999%' is not hyperbole for 'almost always' but is the scientifically established value to 5 significant figures.

4. Results Summary

We have experimented on numerous patience games. Our results fall into three categories: those for games already studied, those for main games which have not been studied before, and finally an extensive investigation into how varying the rules of *Klondike* affects winnability. We provide summary of our winnability estimates for each game in the following three subsections. Details of how these results were obtained occupy the bulk of the rest of this paper. Our focus in this paper has been on winnability, rather than accurate measures of time used for benchmarking purposes. However, we provide summary data of time used and nodes searched in the experiments reported in this paper in Table 10, page 51. Overall, the experiments reported in this paper used about 30 years of CPU-time.

4.1 Results for Previously Studied Games

Solvitaire is able to solve a wide range of previously-researched games, although we have not extended it to be able to search every game that has already been studied. Results are shown in Table 1, page 7. In most cases we improve on previous results, and in some very famous games the improvements are dramatic. For example, we have improved the 95% confidence interval for both *Klondike* and *Canfield* by a factor of 30 over the previous best known results. We have also used Solvitaire to identify bugs in previous solvers for those two games: see Section 7.1. All results except for *Gaps (One Deal)*, *Spider*, and two variants of *Klondike*, have a 95% confidence interval within ±0.1%, and this is the first time this has been achieved for ten of these games. There are games where Solvitaire is not at good as existing solvers, as we discuss further in Section 9.

4.2 Results Only Obtained Using Solvitaire

The second class of results is those on which Solvitaire is responsible for the only good estimate of winnability that we know of. For new results, we have limited our presentation of results to those for which we can give a very small confidence interval. In Table 2, page 8, we give results for 20 games we experimented on ourselves, including some variants that we invented for the purposes of this paper to illustrate the flexibility of our rule language. Most games we give new results for are single-deck games, but we do report a good estimate for the two-deck game *Mrs Mop*. Additionally, Table 2 shows results for two variants of *Carpet*, for which the JSON rules were constructed and experiments performed by Masten (2022a).

All but one of the results shown in Table 2 have a 95% confidence interval within ±0.1%: the exception is *Streets and Alleys*, for which the number of unknown results limited us to ±0.2%.

One interesting game not included in Table 2 is one we invented based on Parlett's (1980) game *Black Hole* with the addition of one free cell: we call the game "Worm Hole". Using Solvitaire, we gave the first good estimate of winnability in an earlier version of our paper, but these results have now been improved on by Masten (2022d), as shown in Table 1. Interestingly, that improvements comes from the Masten's use of a game-specific dominance we were not aware of.

Among the games we study is a stricter variant of thoughtful *Canfield* (invented for this paper) in which moves of partial piles are not allowed: our results show that about 3.7% of games are winnable with the weaker rules but cannot be won with the stronger.

Game	Variant	Solvitaire 95% C.I.	Best Other 95% CI	Citation
Accordion	[*Th.*]	99.99948 ± 0.00052%	**99.9999936 ± 0.0000064%**	Masten (2022c)
Baker's Game		**75.053 ± 0.028%**	**75.011 ± 0.028%**	Pringle (2017, 2018)
				Note: Using a solver by Shlomi Fish
Black Hole		**86.944 ± 0.022%**	86.986 ± 0.053%	Masten (2022c)
Canfield	[*Th.*]	**71.245 ± 0.031%**	*71.872 ± 1.059%*	Wolter (2013b)
				Note: For discussion of Wolter's code, see Section 7.1
Eight Off		99.8805 ± 0.002%	**99.8801 ± 0.0010%**	Masten (2022c)
Fore Cell		**85.617 ± 0.024%**	85.605 ± 0.385%	Keller (2015)
				Note: Michael Keller reports results obtained by Danny A. Jones
– " –	Same Suit	**10.564 ± 0.020%**	10.556 ± 0.061%	Masten (2022c)
				Note: Fore Cell (Same Suit) is the same game as Eight Off (4 Depots)
FreeCell		99.998881 ± 0.000207%	**99.998812 ± 0.000008%**	Fish (2018)
– " –	0 Cells	0.2137 ± 0.0031%	**0.2173 ± 0.0012%**	Fish (2021)
– " –	1 Cell	**19.348 ± 0.093%**	19.519 ± 0.291%	Keller (2015)
– " –	2 Cells	**79.544 ± 0.091%**	79.468 ± 0.126%	Fish (2012)
– " –	3 Cells	**99.3583 ± 0.0162%**	**99.3608 ± 0.0167%**	Keller (2015)
– " –	4 Piles	**0.00866 ± 0.00058%**	0.02162 ± 0.01496%	– " –
– " –	5 Piles	**3.859 ± 0.040%**	3.996 ± 0.248%	– " –
– " –	6 Piles	**61.421 ± 0.098%**	61.719 ± 0.738%	– " –
– " –	7 Piles	**98.857 ± 0.023%**	98.875 ± 0.119%	– " –
Gaps	One Deal	**85.815 ± 3.717%**	89.310 ± 9.124%	Helmstetter and
– " –	Basic Variant	**24.902 ± 0.028%**	*24.809 ± 0.847%*	Cazenave (2004)
				Note: For Basic Variant, raw results unstated in cited paper.
Golf	[*Th.*]	**45.109 ± 0.032%**	45.077 ± 0.309%	Wolter (2013b)
King Albert		**68.542 ± 0.092%**	71.189 ± 8.678%	Roscoe (2016, 2019)
Klondike	[*Th.*]	**81.945 ± 0.084%**	*84.175 ± 2.998%*	Birrell (2017)
– " –	Draw 1	**90.480 ± 0.116%**	*92.589 ± 2.545%*	– " –
– " –	Draw 2	**88.620 ± 0.135%**	*91.213 ± 3.121%*	– " –
– " –	Draw 4	**69.337 ± 0.098%**	*71.111 ± 3.102%*	– " –
– " –	Draw 5	**53.434 ± 0.099%**	*52.640 ± 3.139%*	– " –
– " –	Draw 6	**35.854 ± 0.095%**	*34.559 ± 2.942%*	– " –
– " –	Draw 7	**23.779 ± 0.084%**	*23.402 ± 2.618%*	– " –
				Note: For discussion of Birrell's code, see Section 7.1
Late-Binding Solitaire		**47.021 ± 0.032%**	45.418 ± 3.081%	Ross & Knuth(1989)
Seahaven Towers		89.332 ± 0.020%	**89.319 ± 0.016%**	Masten (2022c)
				Note: Using a solver created by Don Woods
Simple Simon		**97.450 ± 0.034%**	94.910 ± 5.090%	Fish (2009)
Spider	[*Th.*]	98.487 ± 1.513%	**99.9886 ± 0.0114%**	Robinson (2020)
				Note: Literature results mainly computer-solved but some human-solved
Thirty Six	[*Th.*]	**94.674 ± 0.100%**	94.488 ± 0.307%	Wolter (2013b)
Trigon		**15.996 ± 0.023%**	16.008 ± 0.073%	Wolter (2013b)
Worm Hole		99.8886 ± 0.0074%	**99.8906 ± 0.0065%**	Masten (2022c)

Table 1: Comparison with previous work using a consistent methodology for calculating 95% confidence intervals (CI) described in Section 8.1. For numbers used for calculation of 95% CI values, see Table 10 (for Solvitaire) and Table 7 (for data from the literature). State-of-the-art results for each game are in bold. Italics indicate results from the literature open to doubt, see accompanying note. [*Th.*] Thoughtful variant where position of all cards known at start.

Game	Confidence Interval Percentage Range		
Alpha Star	47.794%	±	0.032%
American Canister	5.606%	±	0.015%
Beleaguered Castle	68.170%	±	0.099%
British Canister	0.000129%	±	0.000008%
Canfield (Whole Pile Moves) [$Th.$]	67.562%	±	0.034%
Carpet [$Th.$]†	87.558%	±	0.021%
– ” – (Pre-founded Aces) †[$Th.$]	95.186%	±	0.014%
Delta Star	34.413%	±	0.030%
East Haven [$Th.$]	82.844%	±	0.100%
Fan	48.776%	±	0.099%
Fortune's Favor [$Th.$]	99.9999879%	±	0.0000022%
Mrs Mop	97.992%	±	0.079%
Northwest Territory [$Th.$]	68.369%	±	0.094%
Raglan	81.226%	±	0.085%
Siegecraft	99.136%	±	0.020%
Somerset	53.725%	±	0.097%
Spanish Patience	99.863%	±	0.003%
Spiderette [$Th.$]	99.620%	±	0.018%
Streets and Alleys	51.187%	±	0.186%
Stronghold	97.379%	±	0.042%
Thirty	67.454%	±	0.030%
Will O' The Wisp [$Th.$]	99.9240%	±	0.0027%

Table 2: Solvability percentage: estimates of 95% confidence interval for patiences which were obtained for the first time using Solvitaire. † *Carpet* experiments were performed by Masten (2022a) with results shown in Table 7. Other experiments performed by us have results shown in Table 10. [$Th.$] Thoughtful variant where position of all cards known at start.

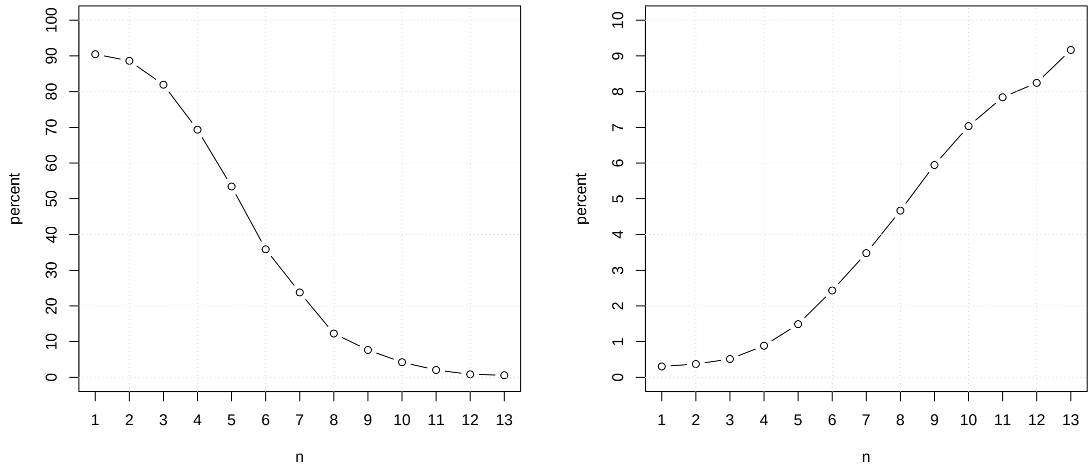

Figure 2: *Klondike* variations: draw size from stock (x-axis) plotted against (left) percentage winnability and (right) percentage of winnable games for which worrying back is critical

4.3 Results on Variants of Klondike and Freecell

As well as their general comment on the embarrassment of not knowing the winnability of *Klondike*, Yan et al. (2005) also commented that "simple questions such as ... *How does this chance depend on the version I play?* remain beyond mathematical analysis". Solvitaire's excellent performance and flexible rule-language gives an ideal framework to study this question. We studied a number of variants of the rules of *Klondike* to investigate how winnability of the thoughtful game was affected. As with our general results, we undertook both replications/improvements and new studies.

As a replication, we also experimented on a number of variants of *FreeCell* that have previously been experimented on, with results in Table 1, page 7. All results are consistent with previous work, with overlapping estimates of confidence interval. Several are significant improvements on knowledge. Table 1 also compares our own results with Birrell's (2017) reported results for *Klondike* with varying draw sizes, and our results represent significant improvements.

For new studies we performed an extensive study of variants of *Klondike*. An important aspect of this study was to reuse results for one game on related games, as described below in Section 6: this greatly reduces the time needed to conduct such large sets of experiments.

Table 3 and Figure 2 show the results on varying numbers of draw size combined with whether or not **'worrying back'** is allowed. As well as seeing the decline of solvability with increasing draw size, we also see the increasing importance of worrying back. By the largest draw size more than 9% of winnable games require it at least once. This is to be expected, as the reduced percentage winnability correlates with fewer routes to win, meaning that there are fewer ways to avoid worrying back.

We also experimented on varying some of the core rules of *Klondike*, specifically what is allowed to be placed in spaces and which suits are allowed for building piles. Table 4 shows the results on *Klondike* with draw size 3 and nine combinations of rules. We see that rules can be significantly more effect when combined than individually. For example, from the most liberal rules (top-left), restricting spaces to kings reduces winnability by only 0.068% and changing the build policy to red-black reduces winnability by 4.964%. However, combining the two restrictions reduces winnability by 17.978%.

Klondike	Worrying Back		
Draw Size	Allowed	Not Allowed	% Critical
1	90.480 ± 0.116%	90.204 ± 0.093%	0.31%
2	88.620 ± 0.135%	88.289 ± 0.112%	0.37%
3	81.945 ± 0.084%	81.524 ± 0.089%	0.51%
4	69.337 ± 0.098%	68.723 ± 0.095%	0.89%
5	53.434 ± 0.099%	52.638 ± 0.099%	1.49%
6	35.854 ± 0.095%	34.982 ± 0.094%	2.43%
7	23.779 ± 0.084%	22.952 ± 0.083%	3.48%
8	12.276 ± 0.065%	11.703 ± 0.064%	4.67%
9	7.670 ± 0.053%	7.214 ± 0.051%	5.95%
10	4.237 ± 0.040%	3.939 ± 0.039%	7.03%
11	2.066 ± 0.029%	1.904 ± 0.027%	7.84%
12	0.849 ± 0.019%	0.779 ± 0.018%	8.24%
13	0.600 ± 0.016%	0.545 ± 0.015%	9.17%

Table 3: Results for *Klondike* with varying draw size and whether or not worrying back is allowed. The final column shows the percentage of winnable games with worrying back that cannot be won without using it least once.

5. Exhaustive Search using AI Methods in Solvitaire

Solvitaire is a depth-first backtracking search solver over the state space of legal card configurations. For good performance, we needed to improve many aspects of the search procedure from this minimal description, and we use a number of techniques from Artificial Intelligence (AI) to do so. We do not claim novelty for these improvements, as many have been applied before to patience solvers, singly and in various combinations (Wolter, 2014d; Birrell, 2017), but their use in combination in a very general patience solver is novel. In this section we describe relevant aspects of design decisions in Solvitaire and use of AI search

Build Policy / Spaces Policy	Any Suit	Red-Black	Same Suit
Any	99.923 ± 0.006%	94.959 ± 0.045%	40.762 ± 0.097%
King Only	99.855 ± 0.049%	81.945 ± 0.084%	6.895 ± 0.050%
Not allowed	*51.135 ± 48.759%*	2.168 ± 0.121%	0.178 ± 0.009%

Table 4: Our winnability estimates on variants of standard *Klondike* with draw size 3 and worrying back allowed. Rules vary on how cards can be built on in the tableau, and what cards if any may be placed into a space. Standard Klondike is in the central cell. The entry in italics for Not Allowed/Any Suit is as computed by our protocol but is not a useful confidence interval.

techniques. After describing our use of depth-first search in Section 5.1, we then describe our use of transposition tables in Section 5.2, symmetry in Section 5.3, dominances in Section 5.4, and streamliners in Section 5.5.

The optimisations were included in Solvitaire following informal investigation and experimentation during the design process. To give an illustration of their effectiveness in a key game, throughout this section we compare how each optimisation affected behaviour in *Klondike*. Table 5, page 14, shows results of Solvitaire with various optimisations enabled or disabled. While we give a number of performance indicators, the most important for us is the number of instances that could be correctly resolved within one hour since we wished to avoid instances that cannot be resolved. Experiments for Table 5 were run on the Cirrus HPC system. CPU Nodes contain 2×Intel Xeon "Broadwell" 18-core cpus, 2.1 Ghz, and 256 GB RAM. We tested the same 10,000 instances of *Klondike* in each configuration so performance is directly comparable.

5.1 Exhaustive Depth-First Search

A key, early, design decision was to prioritise the ability to determine with certainty whether a given instance of a game is winnable or unwinnable. A consequence of this was the decision to optimise for efficient *exhaustive* search for unwinnable instances, with less effort devoted to finding solutions quickly. This led to the choice of depth-first search since it can be implemented extremely efficiently with very little overhead per node searched.

The core depth-first search process is as follows at each node in search, starting with the root node being the initial position Solvitaire has been given. From the initial position, all possible legal moves are constructed and then one chosen for exploration. This is repeated at each new position. If a position is reached where the game has been won, then search is finished. Alternatively, if no legal moves to a new position are possible, then search backtracks to the last parent of this position and tries an alternative move at that parent. This will be one of the other possible legal moves at this node previously constructed. If this process eventually exhausts the possibilities for the starting position, then the instance is proven to be unwinnable. Because this process yields complete exhaustive search, if Solvitaire reports that an instance of a game is unwinnable then it has explored every possible way this could be done, so the statement will be correct. Except for games which are very nearly 100% winnable, obtaining accurate estimates of winnability requires this kind of certainty.

For efficiency, we use trailing instead of copying to save and restore state in backtracking (Schulte, 1999). Specifically, we keep a single full copy of the search state which is subject to change when each move is made. At each node in search we store what move is made. Each move is reversible, so when we backtrack the move is reversed to produce the same state as before.

Although not a general AI technique, we use the K^+ representation of stock (Bjarnason et al., 2007) for patience games with stocks in which there are infinite redeals. This has proved to be an important optimisation in games such as *rollout1,Klondike* and *Canfield*. This replaces the concrete moves which move the stock cards (e.g.) three at a time, with calculating which cards can be obtained next using any sequence of individual stock moves. While it increases the branching rate, it also reduces search depth and ensures that each stock move makes concrete progress instead of just moving cards around pointlessly.

One point to note is that we do not implement any equivalent of 'supermoves' (Keller, 2015) where a sequence of several independent moves are combined in a single step to achieve something useful not possible in one move in the normal rules of the game. There is no need to do this for determining winnability: while supermoves can be useful to players, all possibilities have to be considered in exhaustive search. Including them would make the search process more complex and error-prone to implement, without necessarily obtaining any search advantage.

A remarkable feature of some games is the extraordinary depths that search can reach while still being successful. For example, in Beleaguered Castle, one game was solved at a search depth of more than 190 million, while another was proved unwinnable with a maximum depth of more than 27 million. The latter case involved a total search of just over 1 billion nodes, so an average of less than 40 nodes per search depth. Notice that this remains a complete exhaustive search, so while it could accurately be described as going down a very deep rabbit hole, it was still able to completely explore the entire rabbit warren. This indicates an unusual search space. It remains open whether understanding this unusual behaviour can help improve search.

The choice of depth-first search has been very successful, as shown by results in this paper, but it does result in some tradeoffs. We would mention two in particular. First, we do not even approximate getting the shortest possible solution: in the example above there may well be a solution at depth 190 instead of 190 million. Second, search can spend a long time in an area where there is no solution after an early incorrect choice, leading to very long search times for instances that might be easily solvable by more flexible methods. We did consider the use of iterative deepening to avoid the first problem and also possibly the second. However, preliminary experiments suggested that the overhead of iterative deepening did not pay off for our primary goal of determining winnability.

5.1.1 Configurable Rule-Sets

Listing 1: Rules of *Klondike* in our JSON format. Note the specification of a dominance in moving built groups to limit the available moves, as discussed in Section 5.4.2.

```
"tableau piles": {
     "count": 7,
     "build policy": "red-black",
     "spaces policy": "kings",
     "move built group": "partial-if-card-above-buildable",
     "diagonal deal": true,
     "face up cards": "top" },
"foundations": {
     "removable": true },
"stock": {
     "size": 24,
     "deal count": 3,
     "redeal": true }
```

An important feature of our solver is that games are not hard-wired into the solver. That is, the input to the solver is a description of the rules of the game in a textual format in JSON, (Crockford, 2006) specifying values for different aspects of the game. As an example, the rules of *Klondike* in this format are shown in Listing 1. While games like *Klondike* are provided by name for convenience to the user, this simply means that the JSON is included in the executable rather than preprocessed in any way. Configurable rule-sets also enables us to alter the rules of existing games to test how they affect the winnability of the game, as we showed in Section 4.3. However, our rules language does not cater to every possible patience: we were concerned that a much richer language might have made search less efficient. Our chosen tradeoff between expressiveness and complexity enabled us to obtain many new and improved results. In Appendix F we give the full JSON schema for the rules language, as well as the default values which are used unless overridden.

The use of a flexible rule description language gives us two huge advantages over all previous work in the area, which has allowed at most a limited flexibility of game definition within a relatively small family. The most obvious advantage is the wide range of games that can be experimented on without any adaptation at all of the underlying search engine. This can be seen throughout this paper, where we experimented on dozens of very different games, as well as many minor variants of some. Games that we had not considered at all can be tested just by constructing appropriate JSON input: Masten (2022a) did this to use Solvitaire to find the winnability of two variants of (thoughtful) *Carpet*. The second advantage is that, when we fixed bugs or introduced optimisations for a particular rule, all games using that rule gained the advantage of improved results. For example, the dominance we prove in Appendix C.2 has previously been used only in special-purpose solvers for *Canfield* and *Klondike* but could be applied without change to *Northwest Territory*, where it massively improved our ability to solve this game. As well as greater efficiency this enhances robustness of our results, since any remaining bugs for a given rule will have had to escape detection in any game they applied to.

We use a very naive approach to create the list of possible legal moves at each node in search. Apart from processing the JSON rules for a game into an internal data structure, we do not optimise checking which rules apply. Solvitaire exhaustively checks possible game rules to find which are being used in the current game, and then whether any lead to possible legal moves in the current position. This naive approach does have potential inefficiencies. For example if a game does not contain free cells this fact is checked at each node in search instead of just once at the root. We also do not have any preprocessing or preserve sets of legal moves between states except that having computed the legal moves we retain the list for possible backtracking. At each new node, we simply compute this list from scratch. It was a surprise to us that this very straightforward approach was still so effective in practice, but it is certainly possible that it could be optimised to give even better results.

5.2 Transposition Tables

We use transposition tables (Greenblatt et al., 1967; Smith, 2005) to avoid trying the same position twice. To do this we record every attempted position in a cache. Any position we might consider which is already in the cache can be ignored: its existence in the cache means that it would be potentially explored twice. Akagi, Kishimoto, and Fukunaga (2010) show that the use of transposition tables can lead to suboptimal solutions, but this is not an issue for us as our design goal was simply to find any solution rather than optimal ones.

Some care is needed to ensure that a cache hit correctly links to a previously explored position, so it is important to ensure that a complete game state is stored in the cache. For example, if the cache does not record whether cards in the layout are face-down or not, then obscure bugs can result. We never need to retrieve any data from the cache except the existence of the state, so to save space we store a compressed representation of the state. For each component of the layout (stock, tableau pile, etc.) the cards in that component are listed in order. Also, we need to take care in recording points such as which cards are hidden and face-up. This is not a highly optimised representation but is much smaller than the representation used for the active state. A secondary use of transposition tables is to avoid loops, i.e. a sequence of moves which arrives in a state previously visited as a parent of the current node. This actually reduces to the same case as the general one.

	Number			All Determined				Winnable		Unwinnable	
	>1hr	Solved		Knodes	cpu(s)	RAM mean	RAM max	num	Knodes	num	Knodes
Cache Size											
1,000,000	434	9,566		8,967	19.12	29.5	354.0	7,992	3,992	1,574	34,230
2,000,000	337	9,663		8,046	17.12	47.2	706.6	8,035	3,728	1,628	29,360
5,000,000	241	9,759		6,623	14.50	84.4	1,740	8,060	2,095	1,699	28,100
10,000,000	169	9,831		6,442	14.25	131.3	3,474	8,086	3,093	1,745	21,960
20,000,000	113	9,887		6,310	14.14	200.9	6,933	8,104	2,822	1,783	22,160
50,000,000	65	9,935		6,331	14.68	324.1	17,180	8,118	2,928	1,817	21,530
100,000,000	38	9,962		7,764	18.42	440.6	34,010	8,123	3,141	1,839	28,180
200,000,000	21	9,979		9,628	22.86	558.0	67,500	8,127	3,618	1,852	36,000
Symmetry											
Full	38	9,962		7,764	18.42	440.6	34,010	8,123	3,141	1,839	28,180
None	195	9,805		16,880	29.19	786.1	34,410	8,068	10,460	1,737	46,720
Dominance(s)											
None	481	9,519		33,240	63.11	1,331	34,450	7,908	26,920	1,611	64,260
Safe foundation moves	294	9,706		28,710	57.79	1,066	34,360	8,004	20,830	1,702	65,760
Partial pile restriction	92	9,908		10,620	25.30	652.9	34,280	8,109	4,685	1,799	37,360
Both	38	9,962		7,764	18.42	440.6	34,010	8,123	3,141	1,839	28,180
Streamliner(s)			×								
None	38	9,962	0	7,764	18.42	440.6	34,010	8,123	3,141	1,839	28,180
Foundations	13	9,987	120	6,416	14.22	346.7	33,840	8,007	2,723	1,980	21,350
Suit	0	10,000	1	3,161	7.691	179.2	33,910	8,130	980.2	1,870	12,640
Foundations+Suit	0	10,000	126	2,355	5.280	133.5	33,070	8,005	871.2	1,995	8,309
Smart	33	9,967	0	6,883	15.84	368.6	37,660	8,128	939.4	1,839	33,150

Table 5: **Results of variants of Solvitaire on the same 10,000 instances of *Klondike*.**
Number incomplete in one hour are shown first, and these are not included in other statistics. Mean cpu time (seconds), mean number of nodes searched (in kilonodes, i.e. thousands of nodes), and mean and maximum RAM used (in MB) are given for all determined instances. Number winnable/unwinnable is also given, with mean nodes taken for each category. The column marked × is only relevant to streamliners: it indicates how many winnable problems the streamliner incorrectly reported as unwinnable. All results are to 4 significant figures.
In each family results for the following base setting is repeated and indicated in bold: a cache limited to 100,000,000 entries; the use of both the dominance which force moves to foundations when safe and the dominance which limits moves of partial built piles; the use of full symmetry in considering cached states; and the use of no streamliner.

If the transposition table becomes full, we discard elements on a least-recently-used basis. The exception is that we never discard any ancestor of the current state, as otherwise loops can occur. If the transposition table is entirely full and all states in it are ancestors, then we give up on search and report that a memory-out has occurred. In extreme cases very large amounts of RAM are necessary, up to hundreds of gigabytes of RAM in some of the hardest problems we solved.

The first set of experiments in Table 5 shows how performance varies with size of transposition table. Increasing cache sizes give better results, and we see no point of diminishing returns in our experiments. With a one million sized cache more than 4% of instances remained unresolved, while with the largest size of 200 million, this reduced to 0.2%. However, this improvement does come at considerable space cost. As would be expected, we see the RAM usage increase linearly with the size of cache. While the mean usage remains reasonable, the worst case with the largest cache was a requirement of 67GB. In our experiments, this limited the number that could be run simultaneously on a single machine. The conclusion seems clear, that one should use the largest cache that is consistent with the resources available. It also suggests that a more highly optimised cache representation could lead to better results by using less memory.

5.3 Symmetry

Symmetry in search problems has often been pointed out as an issue which can lead to much redundant search (Gent et al., 2006). That is the case in patience games where we can have equivalent but non-identical positions. A common example in patience games is that all spaces in the tableau are equivalent. We should not waste time trying a card in a second space if it did not work in the first. The use of symmetry is also related to transposition tables, because it means that a single cached state can represent many future states in the game. This is because layouts which differ only in the order of piles are considered identical. More subtly, if a sequence of moves precisely swaps two complete piles from an original position, then we should stop search as we have just returned to an equivalent position. We take a simple but effective approach to avoid this problem. Before storing states in a cache we reduce them to a canonical form, maintaining each group of indistinguishable locations such as tableau piles and free cells in a sorted order. For efficiency this order is maintained incrementally during search. Additionally, where a game does not use suits in any way (for example *Black Hole*) the canonical form can discard suit information for greater reduction.

Choice of when to use symmetry breaking techniques can be handled automatically. None of our rules allow distinction between different tableau piles, free cells etc, so these can be safely assumed to be indistinguishable. On the other hand, whether or not suits are indistinguishable depends on the rules of the game. But the rules language (Appendix F) can be checked for components which depend on suit, such as building to foundation or building within the tableau. If no rules do have this dependency then symmetry can be added to the use of the transposition table. Table 5 clearly shows that switching symmetry off increases the number of unresolved instances five-fold. In this case there is no RAM penalty compared to use of transposition tables alone, so this is an unambiguous win.

5.4 Dominances

The use of ‘dominances’ has proven to be important in AI search (Chu & Stuckey, 2015). A dominance occurs when we can commit to not considering some legal transition in the search space, having detected that a solution in which we make that transition is 'dominated' by an alternative sequence in which we do not make the transition. As an example from 1962, the 'pure literal' rule in the classic DPLL algorithm is a dominance (Davis, Logemann, & Loveland, 1962). The general idea has been widely used in search problems in many areas of AI, for example as stubborn sets in verification (Valmari, 1991), partial order reduction in planning (Wehrle & Helmert, 2012), and automatic move pruning in single-player games (Burch & Holte, 2011).

There are two key types of dominances in searching patience games. First, a move which we can commit to making in a given situation and therefore avoid backtracking from the choice, because we know that if any solution exists, there is a solution where this move is made next. Second, a possible move which we can decide not to attempt at all, because we know that if that move leads to a solution, there is another way of winning the game without making that move next.[8]

Although not under that name, previous workers on patience have recognised the importance of dominances since they can greatly reduce the search space to explore (Keller, 2012; Masten, 2022d; Wolter, 2013a; Birrell, 2017). However, there are some issues with the use of dominances. First, it can be easy to be misled into thinking some proposed dominance is correct when it can actually lead to bugs. We discuss bugs we found related to dominances in our own and other solvers in Section 7.1. Second, and closely related, dominances have been used without being proven correct including the most widely used. In this paper we therefore give proofs of the two dominances we use, in Appendix C. In Section 5.4.1, we discuss the most commonly used dominance in playing patience games, allowing moves to foundations to be committed to. In Section 5.4.2, we discuss an important dominance which applies to key games like *Klondike* and *Canfield*, and which can greatly improve Solvitaire's performance.

5.4.1 Safe Moves To Foundations

In many patience games, the goal is to move cards to the foundations. Beginners often make such moves whenever possible, but this is not always safe. However, an important family of dominances make these moves when it is genuinely safe to do so, and can thus be used to reduce search.

The most typical games build up by suit on the foundations but build down in alternating colour on the tableau. In such games we can automatically move a card to the foundation if it is at most *two* more than the current card on foundations of the opposite colour and at most *three* more than the current card on foundation of the other suit of the same colour (Keller, 2015, 2012). For example, if the foundations have been built to 8♣, 7♢, 9♡, 8♠, it is safe to build the 10♡ from tableau to foundation unconditionally. The only use we could have for the 10♡ is to put a black 9 on it, which in turn can only be used to put the 8♢ on. But all these cards could instead - and preferably - go to the foundation immediately, so there is no need for them on the tableau and therefore not for the 10♡. Following this, it would not be safe to put up the J♡ to foundation, because we might want to keep it to build down 10♠ and 9♢.[9]

If a game does not allow worrying back, then we can use a slightly stronger rule. The rule as above applies but we can also move to foundation unconditionally if the card is no

8. Because our focus is on winnability of games, we do not insist that the safe sequences be the same length or shorter, so the dominances we use might not be appropriate in searches for the shortest winning sequence.
9. It is unclear where this dominance originated, perhaps being invented independently multiple times. Keller (2012) described it as 'a clear and obvious rule' and states it was implemented in some of the earliest FreeCell programs.

more than one higher ranked than the foundations of the opposite colour (Keller, 2012). The reason is that there are no cards of the opposite colour that can possibly be built in the foundation onto this card. On the other hand, when a game does allow worrying back, then we can add a related dominance. We can ban worrying back from foundations to tableau if the card replaced on the tableau would be eligible for automatic movement to the tableau under the first dominance: such a move would lead to a pointless loop. While seemingly minor, this is important as it ensures that progress is not reversed unnecessarily. This is a slightly stronger and generalised version of a dominance proposed by Bjarnason et al. (2007) for Klondike.

Similar, but less complex, dominances are available with other building rules than the standard red-black. If the build policy is by suit, then we can always require cards to be moved from the tableau to foundations if they can be, since no other card can be built onto them. If the build policy is that building is regardless of suit, then we can move a card to foundation if it is no more than two higher than the lowest card yet built to foundation.

These dominances apply to moving cards from the tableau, as well as from a free cell or the reserve. However, it is not safe to enforce this dominance from the stock - as we discuss in Section 7.1. The exception is when the stock draw size is 1 and infinite redeals of stock are allowed: in this case the stock can be treated as if it were a reserve.

Solvitaire implements all the preceding dominances. While well known, these dominances have not been proven correct. Accordingly we prove them correct in Appendix C.1. All of the preceding discussion concerns single-deck games, since that is what our proof covers: some adjustment to the dominance would be necessary for multiple-deck games.

As well as reduction in search space, when a safe move is available we can save space in the transposition table. If some move would be made by the dominance there is no need to enter the state into the transposition table. We can make all available safe moves and then only store the state when no more are available. If any state reoccurs then the safe moves will be made a second time and the final state at the end of the sequence will be found again.

5.4.2 Tableau Moves of Incomplete Piles

In studying code by Wolter (2014d) for *Canfield* and Birrell (2017) for *Klondike*, we noticed an interesting dominance in both. This is that moves of built piles on the tableau are only allowed if **either** the entire pile is being moved **or** only a part of a pile is being moved and it is possible to build to the foundation the card above[10] the top card in the built pile being moved. Our experiments failed to show any case where this optimisation changed results. Wolter has died and Birrell (2018) did not have a correctness proof. We have not found this optimisation documented in the literature, and its correctness is not obvious, so we give what we believe to be the first correctness proof of this dominance.

In Appendix C.2 we generalise the dominance to make it apply more widely, and then give the detailed proof of correctness. We actually prove a slightly stronger version of the dominance, that the card above must not only be buildable to foundation in principle, but must actually be built to foundation immediately. However, as we had not yet noticed this potential improvement, the weaker restriction as suggested by Wolter and Birrell is what we implemented in Solvitaire's code and experiments we report in this paper. An analogue of the stronger restriction for the game of Worm Hole has proved to be important in effective search (Masten, 2022d). The importance of similar techniques in

10. For clarity, in the built pile 10♣9♡8♠ we say the 10♣ is *above* the 9♡ while the 8♠ is *below* the 9♡. The possible confusion is that 10♣ is placed physically underneath the 9♡ when played on a table.

different circumstances illustrates the need to be able to reason more effectively about dominances in general, to allow them to be used when it is correct to do so, without the need for the detailed kind of proof we presented here.

We can give the intuition behind the dominance which also plays a key role in the proof. Suppose we make a partial pile move but do not immediately build the card above it to foundation. This means the partial pile move was not really urgent so we can delay it until later, or even not do it at all. This is straightforward in all but one case. The exception is where the very next move is to build a different card on the card that we just vacated. We can illustrate by example in the case of a red-black build policy: consider the move of a three card pile 10♣9♡8♠ from the J♢ to J♡, followed immediately by a move of the 10♠ to the J♢. This makes simply delaying the first move impossible as it would invalidate the second move, so we have to take another approach. In this case we cancel the move of 10♣ and change the following move by moving the move 10♠ to the J♡ instead of J♢. This causes no significant problem until we want to build the J♢ to foundation, but it might now be covered by the 10♣ when it was previously free. But if this happens, the J♡ must be free itself by a symmetry argument that the J♢ was originally free to move to foundation. So we can *now* move 10♣ from J♢ to J♡. As required by the dominance, the next move will be of the J♢ to foundation. So we have shown that a game won without complying with the dominance rule can also be won complying with it. Full details of all cases are given in Appendix C.2.

Importantly, we also prove, in Theorem 5, page 44, that the above two dominances are also compatible: i.e. if both apply separately then their combined use cannot lead to incorrect results.

5.4.3 Implementation in Solvitaire

To exploit dominances we adapt the search process when finding and making legal moves. For a safe-tableau move, if any is available then one is made immediately and no alternative moves are stored for backtracking. Also, the state does not need to be recorded in the cache because if the state was revisited then dominance moves would be repeated so it is enough to store the endpoint of a sequence of dominance moves. However we do still record the move for purposes of reversing it later during backtracking. For the incomplete pile dominance, if it applies we do not consider a move legal if it moves a partial pile where the card above cannot be built to foundation. However, we do not force the next move to be of the card above to foundation though Theorem 4 would allow that: this is simply because we did not realise the stronger rule was valid when we implemented Solvitaire.

Dominances that seem correct can easily turn out to be unsafe, as we discuss in Section 7.1. This is a particular problem when using the general rule language such as provided by Solvitaire. Unusual combinations of rules may invalidate a dominance which is valid in very similar games. For the dominance of Section 5.4.2, we require it to be specified in the JSON statement of the rules of the game, avoiding automatically applying it incorrectly. In experiments for this paper we only add the dominance when the game meets the conditions of Theorem 4. We do automate the use of the dominance of Section 5.4.1 but only under strict conditions. First, it is disabled completely for games of more than one deck, for games with *Spider*-type building rules, or games like *Gaps* without either foundations or a hole. Second, the move has to be from tableau, free cell or reserve, with one exception: the exception is that moves from the stock are allowed if there are unlimited redeals and the draw size is 1, since in this case the stock is actually equivalent to a reserve. Finally, when the dominance is allowable, the game rules are checked for what the build policy is and whether worrying-back is allowed. The relevant dominance from Section 5.4.1 is then applied.

Table 5 shows performance with all combinations of investigated the two key dominances that we use. These both prove to be very important: if neither is used we fail to resolve more than ten times as many instances as when both are. While the the partial pile restriction is more critical in *Klondike*, both dominances should clearly be used.

5.5 Streamliners

The final AI technique that we use is 'streamliners' (Gomes & Sellmann, 2004; Wetter et al., 2015). A streamliner imposes an additional property which does not necessarily hold in all solutions. A good streamliner is a property that greatly reduces the search space while also having a good chance of leaving at least one solution. Past patience researchers have used the idea of running a solver which might produce false negatives, thereby speeding up cases where a solution can be found (Fish, 2009), but our implementation generalises this across games.

We use two general streamliners. First, in a game in which cards are moved to foundations, always make such a move when it is possible to do so. This is a very common technique of human players and massively reduces the search space while typically allowing most (but not all) winnable games to be won. When used, this is implemented by treating moves to foundation in a similar way to dominances, making the move immediately when available and not backtracking on this choice. Second, we pretend that cards have more symmetry than they do to increase the chance of cache hits. This is very relevant to games which build down in red-black order on the tableau, but up in suits on foundation. If we have a position that differs from a previously visited state only in suits (but not in colours) in the tableau, it is very unlikely to succeed if the first one does not. Exceptions do occur because of the differences between suits, but again the tradeoff is good for this streamliner. This is implemented in the same way as if the symmetry did in fact apply, by discarding suit information when storing and checking states in the transposition table as discussed in Section 5.3.

In Solvitaire, the user chooses via command-line option whether to use one, both or neither streamliner. This is a run-time option since there are games where streamliners cannot possibly help. If there is a solution found with a streamliner then we have proved the instance is winnable, but if not then we have to start search again without that property holding. This is an example of a multi-phase search, of an incomplete search followed by a complete one, as used for example in the FF planner (Hoffmann & Nebel, 2001). To facilitate this, we provide a command-line option to do this automatically under the name 'smart streamliner'. When this option is used we allocate 10% of the original time-limit for a streamlined search and if that fails to prove the game winnable, we allocate the original time-limit for a search with no streamliner. For many games where it does help, the streamlined search very commonly finds a solution very much faster than the full search would do, leading to greatly improved performance over a large set of instances.

Table 5 shows performance of streamliners on *Klondike*. Notice that both streamliners can yield false negatives, and indeed we find that they do both individually and together, they do greatly reduce runtime. For this reason the best combination is the 'smart' streamliner which first runs for 10% time with both streamliners: unlike a pure streamliner, this overall process cannot give a wrong result. Indeed, we see that smart streamliner gives a slight increase in number resolved but also a more significant improvement in CPU time. It reduces time by a mean of 2.5s per instance, equivalent to about a month of CPU time on our main experiment on a million instances of *Klondike*. In other games we see much more dramatic improvements through streamliners, as shown for example by a more than 40-fold speedup in *FreeCell* (see Table 9 in Appendix D, page 45).

6. Relationships between Games

In some cases one ruleset is **stronger** than another, in that any legal move in the stronger game would also be legal in the weaker one. For example, *Worm Hole* is identical to *Black Hole* but with the addition of a free cell. Any instance that can be won as the stronger game of *Black Hole* must automatically be winnable in *Worm Hole* via the same sequence of moves: Some examples from the rules of *Klondike* further illustrate the concept.

- Allowing worrying back makes a game strictly weaker. If we can win the game without worrying back then we can make the same sequence of moves in the game that allows worrying back.
- Allowing nothing to be put into empty spaces is strictly stronger than allowing only kings to put in spaces, which in turn is strictly stronger than allowing any card to be put in spaces.
- Allowing building down in any suit is strictly weaker than building down in red/black ordering. It is also strictly weaker than building down in same-suit ordering. However, red/black and same-suit orderings are incomparable.
- Any draw size from the stock is strictly weaker than any multiple of it. For example, draw size 2 is strictly weaker than draw size 4 and 6. While draw sizes 4 and 6 are incomparable, they are each strictly weaker than draw size 12.

Where one game is stronger than another, winnable instances of the stronger game must be winnable in the weaker, and unwinnable instances of the weaker game must be unwinnable in the stronger. We can use this to reduce greatly the set of instances that must be tested to obtain results between games.

Despite the extreme amounts of time we spent on the hardest *Klondike* instances, we still found some that were proved unwinnable in weaker games or winnable in stronger ones. Of 396 instances that were not solved directly, 7 were found winnable in games where the stock is drawn in units of 6 instead of 3, and one more where the stock is drawn in units of 9. While valid playthroughs for the original game, the reduced search space in the stronger game allowed the winning moves to be found faster. A further 231 instances were shown to be unwinnable when cards are drawn from the stock in ones. It might seem surprising that it is easier to prove the instance unwinnable in a weaker game. The reason is that drawing cards by one allows for a dominance that is not valid when drawing by 3. Drawing cards by one with unlimited redeals makes all cards in the stock available at any time, so we can apply the dominance described in Section 5.4.1 to the stock as well as to the tableau: this is invalid with other draw sizes. When the dominance is applied it reduces the size of the search space and thus allows all possibilities to be exhausted. This is an example of *relaxation* in a search problem (Hooker, 2002). We were able to use these approaches to resolve 239 of the 396 unknown instances, leaving only 157.

When computing results on two games which were strictly stronger/weaker than each other, we used an identical set of instances in each case. This gives us two significant advantages. First, it acts to reduce the statistical variance in computing the difference in winnability between the games. Second, we were able to exploit the linkage between games to avoid recomputing winnability results we already knew. For example, since draw size 5 is strictly weaker than draw size 10, we did not need to test draw size 10 on any of the 465,656 instances proven unwinnable at size 5. Having found the 42,372 winnable instances with worrying back at draw size 10, these were the only ones we needed to test for winnability without worrying back. We used this to greatly reduce the time taken to compute accurate winnability percentages across a range of related games. This can be seen in Appendix E, which shows for example that we could determine the result of one variant of Klondike on one million instances while testing only 5,997 instances for that specific game. This could be used as an additional form of streamliner when searching for solutions for individual patiences - e.g. when trying to win a game with worrying back one could first try it without, which greatly reduces the search space while often not greatly reducing the chances of winning. This is an interesting area for future research that we have not yet investigated.

Table 4 shows up a weakness in Solvitaire's ability to solve patience games. Of a million instances, more than 97% could not be determined when combining building in any suit with spaces not being fillable. We discuss this weakness further in Section 9.

7. Implementation, Testing and Debugging

Solvitaire is implemented in the C++ programming language. During development, code was profiled to identify hotspots in code which needed optimisation. Some areas which did not turn out to be critical were surprising; for example the code to find available moves is barely optimised despite being used at each node in search.

We used a number of strategies to test our code and reduce bugs to a minimum. First, we used unit, integration and performance tests to guard against regressions in the code. As we introduced new game features we created bespoke, simplified games to target the added functionality. Our tests were build upon these game types, using hand-crafted instances with known solutions. We also had a performance benchmark script, which measured the performance of the solver on a number of benchmark instances to let us know if our latest code changes had slowed it down.

Second, we ran strict and loose versions of particular games over identical instances. Where Solvitaire reported a looser versions of a game as but the stricter version as winnable, a bug was indicated which we then fixed. This can be seen as a form of metamorphic testing, which has also been used in testing constraint solvers which have a similar problem of vast search trees without knowing results in advance (Akgün, Gent, Jefferson, Miguel, & Nightingale, 2018).

Third, we could test our work on the macroscopic scale, by comparing overall results obtained using Solvitaire on games also estimated by previous researchers. For example, we discuss in Section 7.1 that this allowed us to identify and fix a bug in our pseudorandom instance generator. Table 1 shows that, where we were able to compute confidence intervals for related work, all our 95% confidence intervals now overlap with the best existing estimate. Given the complete independence of our implementations with those of many different past researchers, this strongly suggests that bugs that significantly affect winnability percentages are unlikely.

Finally, at the microscopic level, for the games *FreeCell*, *Canfield*, and *Klondike*, we tested individual instances to make sure our solver gave consistent results with independent solvers. For *FreeCell*, we ran Solvitaire on each of the 102,075 unsolvable instances of *FreeCell* found by Fish (2018): all were correctly identified as unsolvable except for two that could not be determined. We tested Solvitaire against the best existing solvers for each of *Canfield* (Wolter, 2014d) and *Klondike* (Birrell, 2017) on 50,000 individual instances each. Detailed study of individual inconsistent results allowed us to determine which solver was correct. If the bug was in Solvitaire, we then corrected it. As discussed in Section 7.1, we also found problems in both existing solvers. Although this happened in very rare cases, it indicates the detailed work that allowed us to discover such rare bugs in existing solvers.

We cannot rule out that bugs remain in our code that might affect winnability of some games, especially using unusual combinations of rules we have not tested exhaustively. Availability of our codebase will enable future researchers to identify any remaining bugs in our code (Blake & Gent, 2019). The code includes random generation of instances which is portable across different machines, so other researchers should be able to recreate the same test instances to check our results against theirs.

7.1 Incorrect Optimisations in Existing Solvers for Klondike and Canfield

We tested Solvitaire's results on 50,000 instances each against the best existing solvers for *Klondike* (Birrell, 2017) and *Canfield* (Wolter, 2014d). Where both the existing solver and Solvitaire determine the answer, they should both agree that a given instance is winnable or unwinnable. Where there was disagreement on a specific instance, we looked at the solution produced by whichever solver claimed the game was winnable, which we could check by hand for correctness. In some cases the inconsistency was due to a different understanding of the rules, in which case we always revised our rules to match those of the existing solver. Some bugs remained, and where the bug was in Solvitaire we corrected it, but some bugs were found in existing solvers.

We discovered the same incorrect dominance in both an earlier version of Solvitaire and in Birrell's (2017) Klondike Solver. This concerned worrying-back, i.e. returning a card from foundation to the tableau. It might seem that it would be unnecessary ever to do this immediately after placing the same card from tableau to foundation, but we can construct instances in which it is necessary. In one such example, we move the 3♣ to foundation, revealing the previously hidden 4♡: the only winning continuation is to reverse this immediately, then move the 2♡ onto the 3♣, uncovering the 5♣ onto which we can now move the pile under 4♡. We believe Klondike Solver incorrectly reports six of its first 50,000 random instances to be unwinnable, due to this or other bugs. Given the much smaller sample of 1,000, we do not know if the results reported by Birrell (2017) are affected.

Wolter (2013a), provided the best previous analysis of *Canfield* giving statistics over 50,000 tests of 35,606 solved, 13,730 proved unsolvable, and 664 indeterminate. The code for his solver is available (Wolter, 2014d). As published, the code gives different results because it implements a rule that only entire columns or the bottom card alone can be moved. This is different from the game rules that Wolter (2014a) gives himself, where partial built piles may be moved instead of just whole columns.[11] A minor change to the published code restores the game to Wolter's (2014a) rules and after doing this we obtained *identical* results to those Wolter (2013a) reported.[12] After making this change, we compared results between Wolter's solver and ours for *Canfield*. There remained discrepancies which revealed Solvitaire to have both an unintended rule and a separate bug. When these were corrected we still found a small number of different results, which led to the discovery

11. Curiously, the implemented rule is precisely that given by Parlett (1980), but we do not know whether this was intentional. We cannot check this since Jan Wolter died on 1 January, 2015. We are happy to have this opportunity to pay tribute to him both for his excellent work on solitaire solving programs, and for his openness in making his code publicly available, allowing us to build on his work.
12. Perhaps Wolter corrected the code but never pushed to Google code, or alternatively computed the results before some later code change.

of two obscure bugs in Wolter's code, arising from an incorrect dominance rule. This was a dominance which forced moves to the foundation be made when an appropriate card was in the last two cards in the stock, because playing these cards could (apparently) never prevent another card being played. Unfortunately, if the number of cards in waste is not a multiple of the number of cards played from stock (typically 3), then immediately playing the last card in stock prevents access to the card at the top of the waste pile, and possibly others. For much more subtle reasons, it is not safe to allow the penultimate card in the stock to be played. While rare, we did see examples of random games where Wolter's code incorrectly reported winnable games as impossible. For example, in one game in which the base card was 5♢, the stock started 3♣ 6♣ 6♢ and ended K♣ 7♢ 5♡ Q♠. There was no solution if the 5♡ (the second last card in the stock) was played immediately. To win, the player has to wait until the 6♢ and 6♣ are both played consecutively. Having delayed the play of 5♡ allows it to be played now, uncovering the 7♢ which can be put on the 6♢. The situation is the curious one that if we have already played 5♡ earlier, then after 6♢ we are able to play either the 7♢ or the 6♣ but not both. We believe a very weak version of Wolter's dominance is correct: when the last card of stock (not the last two) meets the conditions of Section 5.4.1 *and* the stock is currently at a multiple of the draw size, the last card can be moved to foundation. We did not implement this in our code.

To correct these two bugs we rewrote Wolter's code to allow dominance moves only for the last card in stock and only when the number of cards in the waste pile is a multiple of the number of cards played from stock. With these corrections our code does not disagree on any of 50,000 instances we tested. Using this corrected code with the parameters Wolter (2013a) previously used, we obtained 35,605 solved, 13,671 proved unsolvable, and 724 indeterminate instances: if those results had been reported, Table 1 would have a confidence interval of 71.929% ± 1.118% for Wolter's results. This confidence interval did not, at that time, overlap with our results, leading us to investigate closely the pseudorandom generators for both programs. We found flaws in both generators, with Wolter's code producing identical instances on repeated seeds, e.g. the same results for seeds 12 and 1212, and ours a slightly biased sample. We corrected our generator appropriately and our results are now consistent with Wolter's, as shown in Table 1.

The general point we make in this section is not a criticism of other programmers, but to emphasise the ease with which apparently correct optimisations can in fact be wrong, and to show the difficulty that can arise in locating the errors. Additionally, it shows the power of Solvitaire in being able to run such extensive comparisons with other solvers that it is able to find very rare inconsistencies, and the benefit to other games of fixing bugs found while investigating one game.

8. Experimental Methods

8.1 Statistics

Each random instance is necessarily either solvable or unsolvable, and therefore the true picture for any given game is it behaves as a binomial with probability p of success. As discussed in Section 6, in some cases we used winnability facts from stronger or weaker games where this was guaranteed correct, saving considerable time. We used the following consistent protocol for measuring a confidence interval on the estimate of winnability percentage. From a sample, if we know the number of winnable and unwinnable games, we calculate a 95% confidence interval for the true value of p using Wilson's method (Agresti & Coull, 1998). When some instances' winnability are unknown, e.g. due to timeouts, we form the most conservative possible interval by calculating the interval both on the assumption that every unknown instance is unwinnable and on the assumption that every unknown instance is winnable. We then report the range from the lower bound of the first interval to the upper bound of the second. While it would be nice to be less conservative and get a smaller interval, no other totally general approach seems valid: for example, in Spider it is very likely that almost all unresolved instances are winnable, while in *Klondike* most long-running instances turn out to be unwinnable. We normally report percentage winnability to 3 decimal places, but give more places where winnability is very close to either 0 or 100%. We use the most conservative possible rounding: given the number of digits we are reporting, we round the lower bound down and the upper bound up. Given the range calculated, we report it from the centre, plus or minus half the range (with the centre chosen arbitrarily from the two choices where the range is odd in the last digit). For calculating equivalent intervals for comparison with previous work, in most cases we could deduce the raw numbers of solved, unsolvable and indeterminate cases from past publications, and calculate the confidence interval that would result from the same protocol. While a confidence interval we compare against may not be the same as that reported in a previous paper, our comparisons with previous results are on a like-for-like basis without being dependent on varying methodologies for estimating the range of solvability used by different authors. Statistics were calculated using R (R Core Team, 2016).

8.2 Experimental Setup

Monte Carlo methods using pseudo-random generation were used to create instances of each game. We used the Mersenne twister generator (Matsumoto & Nishimura, 1998) mt19337 provided by the C++ standard library to generate a stream of pseudorandom numbers. The stream of numbers passed all tests for randomness in the Dieharder test suite, v3.31.1 (Brown, Eddelbuettel, & Bauer, 2019), simulating the way it was used in our code with the initial seed incremented after every 52 random numbers. We wrote our own code to create instances from the stream of numbers: this generator is portable so should produce identical results for the same seed, and is included in our code for Solvitaire. In running experiments, a critical point is that runs which were unresolved are included in our statistics. In many cases we re-ran failed seeds with larger computational resources, but where we could never resolve the instance, they are included in our data as unknown. It would be improper to ignore them and rerun with a new seed as hard instances can have a different likelihood of being winnable to a new random seed. Having decided on a sample size for an experiment we used a consecutive sequence of seeds for that experiment. Seeds for each instance are recorded in our data. As well as winnability, we recorded many other features of search such as run-time, memory usage, cache usage, and search depth. We do not report those statistics in detail but they are available in full in our data files: Table 10 gives an overview by game of run-time and nodes searched.

Experiments mostly used the Cirrus UK National Tier-2 HPC Service at EPCC (see acknowledgements). Additionally, a small number of our results presented here were obtained on local compute-servers at the University of St Andrews. In selecting experimental parameters such as sample size, number of cores used per machine, timeout limits, and cache sizes, we made choices intended to optimise the computing resources and time available. For example, for some games it was critical to run with very large amounts of RAM, reducing the number that could be run in parallel on one machine. In some cases, we accepted a small number of timeouts in order to get a very large sample size (e.g. *American Canister*). In others where there were many timeouts, we focussed on a smaller sample size but very long runtimes to minimise the number of unknowns (e.g. *Gaps One Deal*).

All results were obtained using Solvitaire, but to save CPU time we sometimes reused results for one game for a related game, as described in Section 6. During experiments, minor changes were made to Solvitaire: our data files indicate the version used for each experiment. As a consistency check, we compared results of the current version (0.10.1) against our reported results by testing 1000 instances (or 100 in two very hard games). In all cases where both versions completed, all features of search including precise numbers of nodes were identical.

9. Evaluation of Solvitaire

We have achieved considerable success using Solvitaire, as we have reported throughout this paper. In this section we reflect on the strengths and weaknesses of our program, Solvitaire, and how future work can build on our success.

A significant feature of our work is that most predecessors have written a special program for each main game, while our single program Solvitaire can solve games from a simple textual description. This gives two significant advantages over previous work. First, the uniform approach enables us to implement advanced AI techniques just once but apply them to many games. Second, we can gain improved confidence in correctness through bugfixes from in one game automatically applying to all others. Therefore our work has significant value even in games where previous studies have been done.

We regard it as remarkable that we have been able to obtain so many new and improved results using a general purpose patience playing program. Normally, we would expect a general purpose program to be significantly outperformed by specially written programs. In some cases we have obtained very much improved results over previous work, but this may be due to the availability of significant computer time on modern CPUs rather than an improved solver. Nevertheless, it remains clear that Solvitaire is an outstanding solver in most games we have evaluated it on.

In Appendix D we compare run times of Solvitaire with the best previous specialised Solvers for *Canfield*, *Klondike*, and *FreeCell*. For Canfield, we found that Solvitaire did not perform quite as well as Wolter's (2014d) solver. For Klondike, we found Solvitaire performed slightly better than Birrell's (2017) solver. For FreeCell, Solvitaire was much worse than Fish's (2024)'s solver when streamliners were not used, but with the use of our smart streamliner did in fact perform slightly better than Fish's, though Fish's solver remains better on the rare unwinnable instances. These comparisons should not be taken as a proper scientific comparison of our solvers with competing ones, since for example other solvers may have options or settings which would improve their performance. It is clear that performance of Solvitaire approaches the performance of state-of-the-art solvers while being much more general.

While giving us many advantages, our configurable rule set has some limitations. For example, it does not allow for games with a fixed number of redeals of stock. We also do not allow for some key rules such as pairing, eliminating games like *Doublets* and many others. These limitations were conscious in the sense that in the expressivity/speed tradeoff, we prioritised getting good performance on games we could express rather than total generality. Another limitation in our design of Solvitaire is that we do not attempt to find shortest solutions to instances, or even solutions with some maximum number of moves.

There are some games where we could not improve on previous results, as seen in Table 1. Some examples of this are very near to 100% winnability, such as *FreeCell*, *Spider*, and *Accordion*. We may have been less effective for these games due to our entirely general approach of prioritising proving whether a game was winnable or not, rather than fastest possible finding of a winning sequence when it existed. A particularly interesting example where another solver outperforms Solvitaire is *Worm Hole*. We invented this game to show the flexibility of our ruleset and got reasonable results, but since doing so Masten (2022d) has reported a dominance we were not aware of which allows for improved performance. This again shows the value of dominances in patience solving, as we discussed in Section 5.4, and how much more remains to be done in this area.

One important weakness we have identified is that in some games, many instances are unwinnable but Solvitaire is unable to prove them so. A particularly clear example of this is seen in Table 4. We believe the problem is the combination of a very liberal rule for moving cards (in this case any suit) with a very strict restriction (in this case unfillable spaces). The liberal rule makes the search space very large, while the restriction means that the location of a small number of cards can make the game unwinnable. In this game, imagine a King covering a Queen of the same suit in the tableau. The Queen can never be reached because the King cannot be moved to a space, while the King cannot be built to the foundation because the Queen is not available. Yet there can be literally billions of potential paths that Solvitaire might have to explore, leading to timing out. We have seen similar problems in other games where a game with many possible moves can be unwinnable for small local reasons, and thrashing occurs. While we did not report results here, we saw this with the game 'Alina' (Gent, 2022), where Solvitaire has never proved one to be unwinnable even though our human examination shows that many cannot be won. It should be possible to create solvers which combine the exploratory search strength of Solvitaire while also adding more reasoning ability to exclude possibilities. For example, one might use an approach for detecting inconsistency in planning problems such as suggested by Bäckström, Jonsson, and Ståhlberg (2013). Some initial work shows that constraint solvers can be used to prove instances unwinnable in Klondike, but much remains to be done (Dang, Gent, Nightingale, Ulrich-Oltean, & Waller, 2024). It is remarkable that Solvitaire has been so successful on so many games despite this weakness.

10. Conclusions

We have shown that a single depth-first search based solver, Solvitaire, is able to produce state-of-the-art results across a very wide variety of patience games. We achieved this by combining a variety of general AI search techniques. In doing so, we have obtained many entirely new results across a wide variety of games. We have also greatly improved the state-of-the-art winnability estimates on many games including some of the most famous games such as *Klondike* (often just called 'Solitaire') and *Canfield*. In a pleasing callback to their origin, we have now used Monte Carlo methods to answer the question that caused Stanislaw Ulam to invent Monte Carlo methods.

Despite the level of interest we described in Section 3, we are surprised that this study is the first of its kind, i.e. an academic study of the winnability of many different patience games. Previous studies within academia have tended to focus one game, possibly with some variants. There have been more wide ranging studies done outside the academic literature, for example by Masten (2022c), Wolter (2013b), and others. Showing that general AI methods can be applied across patience games, we hope very much that other researchers will build on what we have done and no doubt greatly improve on it.

The importance of dominances for patience solving is very high, but a number of significant problems remain with their application, which should be addressed in future work. First, one has to discover dominances in the first place. They can be difficult to notice and are often not well publicised in the literature. There is also the overhead of implementation as they can be quite specialised: for example we have not implemented possible dominances to forbid making pointless moves of a card on the tableau which clears a space that cannot be usefully used. Apart from discovering and implementing dominances in the first place, ensuring their correctness can be very difficult. There is a very close link between streamliners and dominances, since a streamliner is just an incorrect dominance, so unifying their treatment would be interesting. Ideally we would like to be able to apply dominances and streamliners automatically, correctly, and generally. Achieving this remains a key challenge for future work in patience solving.

While we believe we have made a significant contribution to the study of games that have occupied humans for uncounted hours, much remains to be done. Without doubt, the most interesting question we leave open is one we have not attempted to tackle at all. In games with hidden cards like *Klondike*, what is the true probability of winning from a starting position? We have always solved the 'thoughtful version'. When faced with a game of the classic Solitaire, *Klondike*, with no peeking on physical cards or electronic undo button, what is the best attainable probability of winning, and how does one obtain this? For many games, this remains a very hard problem, with an answer that is not currently known for Klondike even within a factor of two. While the thoughtful winnability gives an upper bound, it gives us no direct information about a lower bound.

Data and Code Availability

Full experimental results reported in this paper are available at figshare.com with DOI 10.6084/m9.figshare.8311070.v6 (Gent & Blake, 2024). This dataset includes all runs used to report data in this paper, together with other material such as details of testing and analysis scripts used to compute winnability estimates reported here. The code for Solvitaire is open-source under the GNU GPL Version 2 licence. The code used for this version of the paper is available in Zenodo at identifier doi:10.5281/zenodo.3529524 (Blake & Gent, 2019). Development history of Solvitaire is also available on Github at URL https://github.com/thecharlesblake/Solvitaire.

Author Contributions

IPG proposed and supervised the project. CB and IPG jointly made high-level design decisions. CB made all low-level design decisions, implemented Solvitaire, and named it. CB and IPG debugged Solvitaire, and ran exploratory experiments. IPG ran the full experiments reported here and analysed them. IPG constructed the proof of the Theorems. IPG drafted the paper, with CB and IPG revising it.

Acknowledgements

This work was in part supported by EPSRC (EP/P015638/1). This work used the Cirrus UK National Tier-2 HPC Service at EPCC (http://www.cirrus.ac.uk) funded by the University of Edinburgh and EPSRC (EP/P020267/1).

We thank reviewers of an earlier version of this paper for valuable suggestions for improvement. We thank others who have helped us in our work on patience, including Matt Birrell, Dawn Black, Laura Brewis, Arthur W. Cabral, Gal Cohensius, Nguyen Dang, Shlomi Fish, Jordina Francès de Mas, Alan Frisch, Chris Jefferson, Michael Keller, Donald Knuth, Dana Mackenzie, Mark Masten, Ian Miguel, Peter Nightingale, Theodore Pringle, Bill Roscoe, András Salamon, Felix Ulrich-Oltean, Judith Underwood, Jack Waller, and (posthumously) Jan Wolter.

Ian Gent thanks his mother Margaret Gent (1923-2021) for her patience in teaching him love for the game of patience.

References

Agresti, A., & Coull, B. A. (1998). Approximate is better than 'exact' for interval estimation of binomial proportions. *The American Statistician*, *52*(2), pp. 119–126. doi:10.2307/2685469.

Akagi, Y., Kishimoto, A., & Fukunaga, A. (2010). On transposition tables for single-agent search and planning: Summary of results In Felner, A., & Sturtevant, N. R. (Eds.), *Proceedings of the Third Annual Symposium on Combinatorial Search, SOCS 2010, Stone Mountain, Atlanta, Georgia, USA, July 8-10, 2010*, pp. 2–9. AAAI Press. URL: https://doi.org/10.1609/socs.v1i1.18164, doi:10.1609/SOCS.V1I1.18164.

Akgün, Ö., Gent, I. P., Jefferson, C., Miguel, I., & Nightingale, P. (2018). Metamorphic testing of constraint solvers In Hooker, J. N. (Ed.), *Principles and Practice of Constraint Programming - 24th International Conference, CP 2018, Lille, France, August 27-31, 2018, Proceedings*, Vol. 11008 of *Lecture Notes in Computer Science*, pp. 727–736. Springer. URL: https://doi.org/10.1007/978-3-319-98334-9_46, doi:10.1007/978-3-319-98334-9_46.

Bäckström, C., Jonsson, P., & Ståhlberg, S. (2013). Fast detection of unsolvable planning instances using local consistency In Helmert, M., & Röger, G. (Eds.), *Proceedings of the Sixth Annual Symposium on Combinatorial Search, SOCS 2013, Leavenworth, Washington, USA, July 11-13, 2013*, pp. 29–37. AAAI Press. URL: https://doi.org/10.1609/socs.v4i1.18294, doi:10.1609/SOCS.V4I1.18294.

Birrell, M. (2017). Klondike-Solver. Github Repository. URL: https://github.com/ShootMe/Klondike-Solver.

Birrell, M. (2018). Re: Solitaire Solver. Email to Ian Gent, 18 November.

Bjarnason, R., Fern, A., & Tadepalli, P. (2009). Lower bounding Klondike Solitaire with Monte-Carlo planning. In *ICAPS'09: Proceedings of the Nineteenth International Conference on International Conference on Automated Planning and Scheduling*, pp. 26–33. URL: https://dl.acm.org/doi/10.5555/3037223.3037228.

Bjarnason, R., Tadepalli, P., & Fern, A. (2007). Searching solitaire in real time. *ICGA Journal*, *30*(3), pp. 131–142. doi:10.3233/ICG-2007-30302.

Blake, C., & Gent, I. (2019). thecharlesblake/Solvitaire: Release for Zenodo DOI-issuing (v0.10.2). Zenodo. doi:10.5281/zenodo.3529524.

Brown, R. G., Eddelbuettel, D., & Bauer, D. (2019). Dieharder: A random number test suite. Duke.edu. URL: https://web.archive.org/web/20190804063819/https://webhome.phy.duke.edu/~rgb/General/dieharder.php.

Burch, N., & Holte, R. C. (2011). Automatic move pruning in general single-player games In Borrajo, D., Likhachev, M., & López, C. L. (Eds.), *Proceedings of the Fourth Annual Symposium on Combinatorial Search, SOCS 2011, Castell de Cardona, Barcelona, Spain, July 15.16, 2011*, pp. 31–38. AAAI Press. URL: https://doi.org/10.1609/socs.v2i1.18187, doi:10.1609/SOCS.V2I1.18187.

BVS Development Corporation (2003). Accordion solitaire. Earliest archive 2003, original date unknown. URL: https://web.archive.org/web/20030714125217/http://www.bvssolitaire.com:80/rules/Accordion.htm.

Cabral, A. W. (2019). Seahaven Towers. Email to Ian Gent, 1 September.

Cavendish (1890). *Patience Games*. De La Rue.

Chu, G., & Stuckey, P. J. (2015). Dominance breaking constraints. *Constraints*, *20*(2), pp. 155–182. doi:10.1007/s10601-014-9173-7.

Clarke, M. C. (2009). On the chances of completing the game of "Perpetual Motion". arXiv.cs. doi:10.48550/ARXIV.0907.1955.

Crockford, D. (2006). The application/json media type for JavasSript Object Notation (JSON). RFC 4627, RFC Editor. URL: https://www.rfc-editor.org/rfc/rfc4627.txt.

Dang, N., Gent, I. P., Nightingale, P., Ulrich-Oltean, F., & Waller, J. (2024). Constraint models for relaxed klondike variants In *Modref 2024: The 23rd workshop on Constraint Modelling and Reformulation*. URL: https://modref.github.io/papers/ModRef2024_ConstraintModelsforRelaxedKlondikeVariants.pdf.

Davis, M., Logemann, G., & Loveland, D. W. (1962). A machine program for theorem-proving *Commun. ACM*, *5*(7), pp. 394–397. URL: https://doi.org/10.1145/368273.368557, doi:10.1145/368273.368557.

Droettboom, M. (2023). Understanding JSON schema. Space Telescope Science Institute. URL: https://json-schema.org/UnderstandingJSONSchema.pdf.

Dunphy, A., & Heywood, M. (2003). "Freecell" neural network heuristics. In *Proceedings of the International Joint Conference on Neural Networks, 2003*, Vol. 3, pp. 2288–2293. IEEE. doi:10.1109/IJCNN.2003.1223768.

Eckhardt, R. (1987). Stan Ulam, John von Neumann, and the Monte Carlo method. *Los Alamos Science*, *15*, pp. 131–141. URL: https://web.archive.org/web/20161129030550/https://permalink.lanl.gov/object/tr?what=info:lanl-repo/lareport/LA-UR-88-9068, doi:10.2172/1054744.

Elyasaf, A., Hauptman, A., & Sipper, M. (2012). Evolutionary design of FreeCell solvers. *IEEE Transactions on Computational Intelligence and AI in Games*, *4*(4), pp. 270–281. doi:https://doi.org/10.1109/TCIAIG.2012.2210423.

Fish, S. (2009). Updated Simple Simon statistics. Yahoo! Groups. URL: https://web.archive.org/web/20220428151919/https://fc-solve.shlomifish.org/mail-lists/fc-solve-discuss/archive/0974.html.

Fish, S. (2010). Solving statistics for the first 1 million pysolfc black hole solitaire deals. URL: https://web.archive.org/web/20220805135149/https://www.shlomifish.org/fc-solve-temp/mail-lists/fc-solve-discuss/archive/1034.html.

Fish, S. (2012). Two freecell solvability report for the first 400,000 deals. fc-solve.blogspot.com. URL: https://web.archive.org/web/20130719010443/http://fc-solve.blogspot.com/2012/09/two-freecell-solvability-report-for.html.

Fish, S. (2018). Report: The solvability statistics of the Freecell Pro 4-freecells deals. ShlomiFish.org. URL: https://web.archive.org/web/20180815201227/https://fc-solve.shlomifish.org/charts/fc-pro--4fc-deals-solvability--report/.

Fish, S. (2021). freecell-pro-0fc-deals. Github Repository. URL: https://web.archive.org/web/20220419155553/https://github.com/shlomif/freecell-pro-0fc-deals/blob/master/README.md.

Fish, S. (2024). Freecell solver. URL: http://fc-solve.shlomifish.org/.

Gent, I. P. (2022). Rules of some patience games from "250+ Solitaire Collection". Ian Gent's Blog. URL: https://web.archive.org/web/20220805214221/https://blog.ian.gent/2022/08/rules-of-some-patience-games-from-250.html.

Gent, I. P., & Blake, C. (2024). Patience Experimental Results (Version 6). URL: https://figshare.com/articles/Patience_Experimental_Results/8311070/6, `doi:10.6084/m9.figshare.8311070.v6`.

Gent, I. P., Jefferson, C., Kelsey, T., Lynce, I., Miguel, I., Nightingale, P., Smith, B. M., & Tarim, S. A. (2007). Search in the patience game 'Black Hole'. *AI Communications*, *20*(3), pp. 211–226. URL: https://dl.acm.org/doi/10.5555/1365527.1365533.

Gent, I. P., Petrie, K. E., & Puget, J.-F. (2006). Symmetry in constraint programming. In *Foundations of Artificial Intelligence*, Vol. 2, pp. 329–376. Elsevier. `doi:10.1016/S1574-6526(06)80014-3`.

Gomes, C., & Sellmann, M. (2004). Streamlined constraint reasoning. In Wallace, M. (Ed.), *Principles and Practice of Constraint Programming – CP 2004*, pp. 274–289, Berlin, Heidelberg. Springer Berlin Heidelberg. `doi:10.1007/978-3-540-30201-8_22`.

Greenblatt, R. D., Eastlake, D. E., & Crocker, S. D. (1967). The Greenblatt chess program. In *Proceedings of the November 14-16, 1967, Fall Joint Computer Conference*, AFIPS '67 (Fall), p. 801–810, New York, NY, USA. Association for Computing Machinery. `doi:10.1145/1465611.1465715`.

Helmstetter, B., & Cazenave, T. (2004). Searching with analysis of dependencies in a solitaire card game. In Van Den Herik, H. J., Iida, H., & Heinz, E. A. (Eds.), *Advances in Computer Games: Many Games, Many Challenges*, pp. 343–360, Boston, MA. Springer US. `doi:10.1007/978-0-387-35706-5_22`.

Hoffmann, J., & Nebel, B. (2001). The ff planning system: fast plan generation through heuristic search *J. Artif. Int. Res.*, *14*(1), p. 253–302.

Hooker, J. N. (2002). Logic, optimization, and constraint programming *"INFORMS" Journal on Computing*, *14*(4). `doi:10.1287/ijoc.14.4.295.2828`.

Howe, J. (2006). The rise of crowdsourcing. Wired. June 2006. URL: https://web.archive.org/web/20151028232825/https://www.wired.com/2006/06/crowds/.

Jenkyns, T. A., & Muller, E. R. (1981). A probabilistic analysis of clock solitaire. *Mathematics Magazine*, *54*(4), pp. 202–208. `doi:10.1080/0025570X.1981.11976927`.

Jensen, P. (2020). Celebrating 30 years of Microsoft Solitaire with those oh-so-familiar bouncing cards. Xbox.com. URL: https://web.archive.org/web/20200522210053/https://news.xbox.com/en-us/2020/05/22/celebrating-30-years-microsoft-solitaire/.

Keller, M. (2012). When can I play the six of clubs? Observations on the autoplay controversy. ShlomiFish.org. URL: https://web.archive.org/web/20220823080948/https://fc-solve.shlomifish.org/mail-lists/fc-solve-discuss/archive/1214.html.

Keller, M. (2015). FreeCell – frequently asked questions (FAQ). Solitaire Laboratory. URL: https://web.archive.org/web/20181215222456/http://solitairelaboratory.com/fcfaq.html.

Mackenzie, D., & Graham, R. (2019). Email exchange reported by Mackenzie to Ian Gent, 21 Nov 2019.

Masten, M. (2022a). Carpet. URL: https://web.archive.org/web/20221208175917/https://solitairewinrates.com/Carpet.html.

Masten, M. (2022b). Perpetual motion (narcotic). URL: https://web.archive.org/web/20221208175940/https://solitairewinrates.com/PerpetualMotion.html.

Masten, M. (2022c). Solitaire win rates and analysis. URL: https://web.archive.org/web/20221208175740/https://solitairewinrates.com/.

Masten, M. (2022d). Technical details of Mark Masten's worm hole solver. URL: https://web.archive.org/web/20221208180007/https://solitairewinrates.com/WormHoleTechnicalDetails.html.

Matsumoto, M., & Nishimura, T. (1998). Mersenne twister: A 623-dimensionally equidistributed uniform pseudo-random number generator. *ACM Trans. Model. Comput. Simul.*, *8*(1), pp. 3–30. doi:10.1145/272991.272995.

Parlett, D. (1980). *The Penguin Book of Patience*. Penguin.

Paul, G., & Helmert, M. (2016). Optimal solitaire game solutions using a* search and deadlock analysis. In *Ninth Annual Symposium on Combinatorial Search*, pp. 135–136. URL: https://web.archive.org/web/20220805135304/https://ojs.aaai.org/index.php/SOCS/article/view/18405.

Plante, C. (2012). Unbeatable. The Gameological Society. URL: https://web.archive.org/web/20190516164853/http://gameological.com/2012/04/unbeatable/index.html.

Pringle, T. (2017). Bakersgame-10million. Bitbucket Repository. Accessed 19 September 2018, not archivable. URL: https://bitbucket.org/theodorepringle/bakersgame-10million/.

Pringle, T. (2018). Re: Query about your Baker's Game results. Email to Ian Gent, 25 July.

R Core Team (2016). *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria. URL: https://www.R-project.org/.

Robinson, A. (2020). Winnable spider solitaire games. Tranzoa.net. URL: https://web.archive.org/web/20210305230500/https://www.tranzoa.net/~alex/plspider.htm.

Roscoe, A. W. (2016). Card games as pointer structures: case studies in mobile CSP modelling. arXiv.cs. doi:10.48550/ARXIV.1611.08418.

Roscoe, A. W. (2019). Re: Patience. Email to Ian Gent, 22 August.

Ross, A., & Healey, F. (1963). Patience Napoléon. *Proceedings of the Leeds Philosophical and Literary Society (Literary and Historical Section)*, *10*, pp. 137–190.

Ross, K. A., & Knuth, D. E. (1989). A programming and problem solving seminar. Tech. rep. STAN-CS-89-1269, Stanford University, Stanford, CA, USA. URL: https://web.archive.org/web/20180409232321/http://i.stanford.edu/pub/cstr/reports/cs/tr/89/1269/CS-TR-89-1269.pdf.

Schulte, C. (1999). Comparing trailing and copying for constraint programming In *Proceedings of the 1999 International Conference on Logic Programming*, p. 275–289, USA. Massachusetts Institute of Technology. doi:10.5555/341176.341217.

Smith, B. M. (2005). Caching search states in permutation problems. In van Beek, P. (Ed.), *Principles and Practice of Constraint Programming - CP 2005*, pp. 637–651, Berlin, Heidelberg. Springer Berlin Heidelberg. doi:10.1007/11564751_47.

Valmari, A. (1991). Stubborn sets for reduced state space generation In *Proceedings of the 10th International Conference on Applications and Theory of Petri Nets: Advances in Petri Nets 1990*, p. 491–515, Berlin, Heidelberg. Springer-Verlag.

Warfield, T. (2016a). American canister. Goodsol.com. URL: https://web.archive.org/web/20160324031437/https://www.goodsol.com/games/americancanister.html.

Warfield, T. (2016b). Easthaven. Goodsol.com. URL: https://web.archive.org/web/20160621140912/https://www.goodsol.com/games/easthaven.html.

Warfield, T. (2017a). Northwest territory. Goodsol.com. URL: https://web.archive.org/web/20171010061224/https://www.goodsol.com/games/northwestterritory.html.

Warfield, T. (2017b). Spanish patience. Goodsol.com. URL: https://web.archive.org/web/20170629170622/https://www.goodsol.com/games/spanishpatience.html.

Warfield, T. (2018). Sea towers (Seahaven Towers). Goodsol.com. URL: https://web.archive.org/web/20180520082726/https://www.goodsol.com/games/seatowers.html.

Warfield, T. (2019). Stronghold. Goodsol.com. URL: https://web.archive.org/web/20190122171602/https://www.goodsol.com/pgshelp/index.html?stronghold.htm.

Wehrle, M., & Helmert, M. (2012). About partial order reduction in planning and computer aided verification In *Proceedings of the Twenty-Second International Conference on International Conference on Automated Planning and Scheduling*, ICAPS'12, p. 297–305. AAAI Press.

Wetter, J., Akgün, Ö., & Miguel, I. (2015). Automatically generating streamlined constraint models with ESSENCE and CONJURE. In Pesant, G. (Ed.), *Principles and Practice of Constraint Programming*, pp. 480–496. Springer International Publishing. doi:10.1007/978-3-319-23219-5_34.

Wikipedia Contributors (2017). Beleaguered castle. URL: https://web.archive.org/web/20170205150411/https://en.wikipedia.org/wiki/Beleaguered_Castle.

Wolter, J. (2013a). Experimental analysis of canfield solitaire. Politaire.com. URL: https://web.archive.org/web/20180429220704/https://politaire.com/article/canfield.html.

Wolter, J. (2013b). Experimental analysis of various solitaire games. Politaire.com. URL: https://web.archive.org/web/20170724143426/https://politaire.com/article/intro.html.

Wolter, J. (2014a). Rules for Canfield solitaire. Politaire.com. URL: https://web.archive.org/web/20150218063842/http://politaire.com/help/canfield.

Wolter, J. (2014b). Rules for Thirty Six solitaire. Politaire.com. URL: https://web.archive.org/web/20170619054612/http://politaire.com/help/thirtysix.

Wolter, J. (2014c). Rules for Trigon solitaire. Politaire.com. URL: https://web.archive.org/web/20170618222612/http://politaire.com/help/trigon.

Wolter, J. (2014d). solsolve: a solving workbench for various solitaire games. Google Code Archive. URL: https://code.google.com/archive/p/solsolve/.

Yan, X., Diaconis, P., Rusmevichientong, P., & Roy, B. V. (2005). Solitaire: Man versus machine. In *Advances in Neural Information Processing Systems*, pp. 1553–1560. URL: https://dl.acm.org/doi/10.5555/2976040.2976235.

Appendix A. Rules Of Patience Games

Table 6 shows the rule for most main games in this paper. Exceptional games not in this table are Accordion, Late-Binding Solitaire and Gaps variants, for which detailed rules in JSON are shown in Listings 2 and 3. To present most games in a uniform format, Table 6 uses a very concise notation which is explained below. Where variants of a game are reported in this paper, the rules are as given here except for the stated change, e.g. Fore Cell (Same Suit) has the same rules as Fore Cell except with BP set to =.

Game: Name of game, plus citation which gives to name and rules we use (but is not necessarily the inventor of the game). *Symbols Used:* ∗∗ Game invented for this paper

Decks: Number of complete decks used in the game, of 52 cards by default. *Symbols Used:* ‡ Deck of 32 cards, we use A+2-8 of each suit.

Foundations: Rules for Foundations or Hole. Number of cards initially placed in foundations, • for hole being used, or S for Spider-type elimination of suits. *Symbols Used:* ✓ Worrying back from foundations to tableau is allowed. × Worrying back is not allowed. ¶ Random base of foundation. § A K not considered adjacent in rank.

Tableau: Number of tableau piles, plus shape of tableau. *Symbols Used:* □ piles all of same length except possibly for some piles of one extra length; △ piles in triangular form; solid shapes indicates that cards face-down except the top card, otherwise all cards face-up.

BP: Build Policy, rule by which one card may be placed on another in the tableau. Where allowed, cards must be one lower in rank (including from K on A if Foundations start on random base). *Symbols Used:* × building not allowed; * card of any suit allowed; rb card must be of opposite colour (red on black or black on red); = card of same suit.

MG: Move of Groups, whether or not a consecutive sequence of built cards may be moved as a unit in the tableau. *Symbols Used:* × not allowed; ✓ allowed with the same restriction as BP; = allowed for sequence of cards of the same suit. + Only entire piles may be moved.

SP: What card may be put in a free space in the tableau, or sequence if MG allows it. *Symbols Used:* × Spaces may not be filled; ✓ Spaces may be filled by any card; K Spaces may be filled by a K only (or card one rank below foundation base if random). † Space must be refilled immediately from stacked reserve until that is empty, then may be filled freely. †† Space must be refilled immediately from waste (or stock if empty).

Stock: The first number indicates the number of cards in the stock; the second symbol indicates the number of cards drawn at a time from a stock, with □ indicating that one card from stock is dealt to each tableau pile; the third number indicates whether no redeals are allowed when the stock is empty or an infinite number are.

FC: The number of Free Cells in the game, if any, followed by the number of free cells that are filled at the start of the game.

Reserve: The size of any Reserve in the game. S indicates that the reserve is ‘stacked’, i.e. only the top card of it is available for play. Otherwise all cards are available at any time.

Game	*Rules Citation*	*Decks*	*Foundations*	*Tableau*	TC	BP	MG	SP	*Stock*	*FC*	*Reserve*
Alpha Star	(Gent, 2022)	1	4 ×	12 □	48	=	✓	✓			
American Canister	(Warfield, 2016a)	1	0 ×	8 □	52	rb	✓	✓			
Baker's Game	(Parlett, 1980)	1	4 ×	8 □	52	=	×	✓		4 0	
Beleaguered Castle	(Parlett, 1980)	1	4 ×	8 □	48	*	×	✓			
Black Hole	(Parlett, 1980)	1	• ×	17 □	51	×	×	×			
(British) Canister	(Parlett, 1980)	1	0 ×	8 □	52	rb	×	K			
Canfield	(Wolter, 2014a)	1	1 × ¶	4 □	4	rb	✓	†	34 3 ∞		13 S
Delta Star	(Gent, 2022)	1	4 ×	12 □	48	=	×	✓			
East Haven	(Warfield, 2016b)	1	0 ✓	7 ■	21	rb	×	✓	31 □ 0		
Eight Off	(Parlett, 1980)	1	0 ×	8 □	48	=	×	K		8 4	
Fan	(Parlett, 1980)	1	0 ✓	18 □ ‖	52	=	×	K			
Fore Cell	(Keller, 2015)	1	0 ×	8 □	48	rb	×	K		4 4	
Fortune's Favor	(Parlett, 1980)	1	4 ×	12 □	12	=	×	††	36 1 0		
Freecell	(Keller, 2015)	1	0 ×	8 □	52	rb	×	✓		4 0	
Golf	(Parlett, 1980)	1	• × ¶ §	7 □	35	×	×	×	16 1 0		
King Albert	(Parlett, 1980)	1	0 ✓	9 △	45	rb	×	✓			7
Klondike	(Bjarnason et al., 2007)	1	0 ✓	7 ▲	28	rb	✓	K	24 3 ∞		
Mrs Mop	(Parlett, 1980)	2	S ×	13 □	104	*	=	✓			
Northwest Territory	(Warfield, 2017a)	1	0 ✓	8 ▲	36	rb	✓	K			16
Raglan	(Parlett, 1980)	1	4 ✓	9 △	42	rb	×	✓			6
Seahaven Towers	(Warfield, 2018; Cabral, 2019)	1	0 ×	10 □	50	=	×	K		4 2	
Siegecraft	(Wikipedia Contributors, 2017)	1	4 ×	8 □	48	*	×	✓		1 0	
Simple Simon	(Parlett, 1980)	1	S ×	10 △	52	*	=	✓			
Somerset	(Parlett, 1980)	1	0 ✓	10 △	52	rb	×	✓			
Spanish Patience	(Warfield, 2017b)	1	0 ✓	13 □	52	*	×	✓			
Spiderette	(Parlett, 1980)	1	S ×	7 ▲	28	*	=	✓	24 □ 0		
Spider	(Parlett, 1980)	2	S ×	10 ■	54	*	=	✓	50 □ 0		
Streets & Alleys	(Parlett, 1980)	1	0 ×	8 □	52	*	×	✓			
Stronghold	(Warfield, 2019)	1	0 ×	8 □	52	*	×	✓		1 0	
Thirty	(Parlett, 1980)	1‡	0 ×	5 □	30	*	✓	✓			2
Thirty Six	(Wolter, 2014b)	1	0 ×	6 □	36	*	✓	✓	16 1 0		
Trigon	(Wolter, 2014c)	1	0 ×	7 ▲	28	=	✓	K	24 3 ∞		
Will O' The Wisp	(Parlett, 1980)	1	S ×	7 ■	21	*	=	✓	31 □ 0		
Worm Hole	**	1	• ×	17 □	51	×	×	×		1 0	

Table 6: Detailed rules of main games studied in this paper, excepting those in Listings 2 and 3. See page 33 for key. ** Original game.

Listing 2: Rules of *Accordion*. The rules of *Late-Binding Solitaire* are the same except with size 18..

```
"foundations": {
  "present": false},
"tableau piles": {
  "count": 0},
"accordion": {
  "size": 52,
  "moves": ["L1", "L3"],
  "build policies": ["same-suit", "same-rank"]}
```

Listing 3: Rules of *Gaps (One Deal)*. The rules of *Gaps (Basic Variant)* are the same except with fixed suit true.

```
\begin{alltt}
  "foundations": {
    "present": false},
  "tableau piles": {
    "count": 0},
  "sequences": {
    "count": 4,
    "direction": "L",
    "build policy": "same-suit",
    "fixed suit": false}
```

Appendix B. Summary of Data from the Literature

Results from previous researchers on winnability of patience games is widely spread, and presented in numerous different forms. Here we present the raw data used to generate confidence intervals for existing results throughout this paper. Table 7 shows the raw data for the best results we could find for each game, while Table 8 gives the an archival URL for pages giving the reported data. Archival URLs are particularly important: for example, many results were originally discussed in Yahoo groups, which were deleted in 2020. This list only includes games we have compared with Solvitaire. For other games not included here, useful starting points are the summaries of Keller (2015), Masten (2022c) and Wolter (2013b).

Game		Sample	Winnable	Unwinnable	Unknown
Accordion		3×10^7	30,000,000	0	0
Baker's Game		10^7	7,501,119	2,498,881	0
Note: Fish previously reported 7,431,962 solvable but this used a solver configuration allowing false negatives (Pringle, 2018)					
Black Hole		1.6×10^6	1,391,771	208,229	0
Canfield [*Th.*]		50,000	*35,606*	*13,730*	*664*
Note: We believe some results are wrong: see Section 7.1					
Carpet [*Th.*]		10^6	8,755,758	1,244,242	0
–"– Pre-founded Aces		10^6	9,518,603	481,397	0
Note: Results obtained by Mark Masten using Solvitaire					
Eight Off		5×10^7	49,940,034	59,966	0
Fore Cell		32,000	27,395	4,605	0
– " –	Same Suit	10^6	105,560	894,440	0
FreeCell		8,589,934,591	8,589,832,516	102,075	0
– " –	0 Cells	8,589,934,591	18,577,014	8,571,181,674	175,903
– " –	1 Cell	100,000	19,473	80438	89
– " –	2 Cells	400,000	317,873	82126	1
– " –	3 Cells	10^6	993,600	6,380	20
– " –	4 Piles	32,000	5	31995	0
– " –	5 Piles	32,000	1,266	30,713	0
– " –	6 Piles	32,000	19,685	12,184	131
– " –	7 Piles	32,000	31,641	357	2
Gaps	One Deal	100	88	4	8
– " –	Basic Variant	10,000	*2,480*	*7,520*	*0*
Note: Paper states 24.8% success and sample size, not these precise numbers					
Golf		100,000	45,077	54,923	0
King Albert		100	72	28	0
Klondike [*Th.*]	Draw 1	1,000	*919*	*62*	*19*
	Draw 2	1,000	*801*	*71*	*28*
	Draw 3	1,000	*836*	*149*	*15*
	Draw 4	1,000	*709*	*285*	*6*
	Draw 5	1,000	*526*	*473*	*1*
	Draw 6	1,000	*345*	*655*	*0*
	Draw 7	1,000	*233*	*767*	*0*
Note: We do not know if results are affected by issues discussed in Section 7.1					
Late-Binding Solitaire		1,000	454	546	0
Seahaven Towers		1.5×10^7	13,397,816	1,602,184	0
Simple Simon		5,000	4,533	*0*	*467*
Note: Solver can report false negatives, so unsolvable listed here as unknown					
Spider [*Th.*]		32,000	31,998	0	2
Thirty Six [*Th.*]		100,000	94,327	5,343	330
Trigon		10^6	160,076	839,924	0
Worm Hole		10^6	998,908	1,092	0

Table 7: Totals from the literature used in this paper. Numbers in italics indicate issue discussed in accompanying note. For archival URLs giving source of datas in this table, see Table 8.

Game	Variant	Archival URL
Accordion		20220425085012/https://masten.000webhostapp.com/Accordion.html
Baker's Game		20220425085149/https://masten.000webhostapp.com/BakersGame.html
Black Hole		20220425085046/http://masten.000webhostapp.com/BlackHole.html
Canfield		20180429220704/https://politaire.com/article/canfield.html
Carpet		20220728095714/https://masten.000webhostapp.com/Carpet.html
Eight Off		20220426164250/https://masten.000webhostapp.com/EightOff.html
Fore Cell		20181215222456/http://solitairelaboratory.com/fcfaq.html
– ” –	Same Suit	20220426164250/https://masten.000webhostapp.com/EightOff.html
FreeCell		20180815201227/https://fc-solve.shlomifish.org/charts/fc-pro–4fc-deals-solvability–report/
– ” –	0 Cells	20220419155553/https://github.com/shlomif/freecell-pro-0fc-deals/blob/master/README.md
– ” –	2 Cells	20130719010443/http://fc-solve.blogspot.com/2012/09/two-freecell-solvability-report-for.html
– ” –	Others	20221221122054/https://ipg.host.cs.st-andrews.ac.uk/KellerMillion.htm
Gaps		20180729133856/https://link.springer.com/content/pdf/10.1007/978-0-387-35706-5_22.pdf
Golf		20170625031422/https://politaire.com/article/golf.html
King Albert		20220618044831/https://arxiv.org/pdf/1611.08418.pdf
Klondike		20160218015922/https://github.com/ShootMe/Klondike-Solver/blob/master/Statistics.txt
Late-Binding Solitaire		20180409232321/http://i.stanford.edu/pub/cstr/reports/cs/tr/89/1269/CS-TR-89-1269.pdf
Seahaven Towers		20220426164824/http://masten.000webhostapp.com/SeahavenTowers.html
Simple Simon		20220428151919/https://fc-solve.shlomifish.org/mail-lists/fc-solve-discuss/archive/0974.html
Spider		20210305230500/https://www.tranzoa.net/~alex/plspider.htm
Thirty Six		20170624201624/http://politaire.com/article/thirtysix.html
Trigon		20170625011319/http://politaire.com/article/trigon.html
Worm Hole		20220426164749/https://masten.000webhostapp.com/WormHole.html

Table 8: Archival URLs for sources of data reported in Table 7. Note that URLs are not necessarily those of citations in Table 1. URLs are relative to https://web.archive.org/web/. For the original URL, delete the numerical prefix and first slash.

Appendix C. Proof of Correctness of Key Dominances

The proofs in this section will proceed by permuting the move order, e.g. swapping the order of consecutive moves. For this reason, we will require that moves are not made illegal by their position in the move sequence: so the proof would not apply to a game where moves to foundation could only be made if the position in the move sequence were divisible by 3.

We define a move m as being a pair $[c, t]$ where c is the card being moved and t is the card or location it is moved to. (We assume some unambiguous notation where cards are moved to locations such as spaces which are currently empty.) We write $C(m)$ for the card being moved by a move, and $T(m)$ for the card or location moved to, i.e. if $m = [c, t]$ then $C(m) = c$ and $T(m) = t$. We remark that, in games with multiple decks, there will be distinct cards with the same suit/rank, but these are still considered separate cards.

For the purposes of the proofs, we will call a move 'compliant' if it respects the constraints the dominance places on moves in either disallowing or requiring certain moves, and 'non-compliant' if it does not.

The general pattern of the proofs is to start from any winning sequence of moves in the original rules, containing at least one non-compliant move. From this we will create a new sequence of moves which is both legal and a winning sequence. The new sequence will either have fewer non-compliant moves than the original, or have the same number but with the last non-compliant move closer to the end of the sequence than before. This means that by repeated application of the process we will eventually obtain a sequence of moves that wins the game and has zero non-compliant moves. Thus there is always a compliant sequence available.

C.1 Safe Moves To Foundations

In Section 5.4.1, we described dominances which apply to moving cards from the tableau, as well as from a free cell or the reserve. However, it is not safe to enforce this dominance from the stock - as we discuss in Section 7.1. The exception is when the stock draw size is 1 and infinite redeals of stock are allowed: in this case the stock can be treated as if it were a reserve. While previously described (Keller, 2012), they have not previously been proven correct formally. We give the first such proof in this section.

Definition 1 (Safely buildable)**.**

- *A card c is called '**potentially safely buildable**' to foundation at time i if:*
  - *the card of one rank lower than c and the same suit is already on foundation or card c is the first card to be played to its foundation (usually an Ace);*
  - ***and,** if another card d has c as the target location of any possible future move $[d, c]$ (other than foundation build), then d is also potentially safely buildable at time i.*
- *A card c is called '**safely buildable**' to foundation at time i if it is potentially safely buildable and the move of c to foundation is legal at time i.*

We restrict consideration to games which involve building to foundation and moving all cards there to win. We also assume that we do not have multiple decks: i.e. while the theorem applies to a single deck with eight different suits of two colours, it does not apply to a game with two copies of the standard deck. The occurrence of duplicate cards leads to potential edge cases that we do not consider in this proof. Finally, since our proof involves permuting moves, we assume that the game has no rules in which move order affects a moves legality.

Theorem 1. *In a game of the type described above, a winnable instance is also winnable with the restriction that when any card currently in the tableau/free cell/reserve is safely buildable, the next move must be the move of a safely buildable card to foundation.*

Proof. Consider any winning sequence of moves in the original rules, $m_1, m_2, \ldots, m_n$, containing at least one non-compliant move. Suppose that the move $m_i = (c_i, t_i)$ is the last move in the sequence which is non-compliant. As the last non-compliant move, we know that: $i < n$ since the last move in a winning game must be moving a card to the foundation pile; m_i is not the move of a safely buildable card to foundation; that there must be at least one safely buildable card at time i; and that move m_{i+1} is a compliant move. There are two cases: either card c_i was safely buildable at time i or it was not.

- The first case is that c_i was safely buildable at time i. In the original sequence of moves, c_i was safely buildable at time i and so must have been at the bottom of a built group. It will remain there and so must also be safely buildable at time $i+1$. Since m_{i+1} and all subsequent moves were compliant, there must have been a consecutive sequence of moves from m_{i+1} of safe builds to foundation, and one of these - say m_j - was the first move of c_i to foundation. All these moves remain legal and compliant. We now delete the original move m_i and replace it with m_j, giving a new sequence of moves is $m_j, m_{i+1} \ldots m_{j-1}, m_{j+1} \ldots m_n$. This subsequence has no non-compliant moves as we deleted m_i.

- The second case is that c_i was not safely buildable at time i. First note that move m_i cannot have moved c_i either from or to *any* safely buildable card d. Card c_i cannot have moved *from* d because d was playable to foundation and therefore uncovered. Card c_i cannot have moved *to* d by definition of d being safely buildable: any card movable to d must itself be potentially safely buildable but card c_i was not. Together this means that any safely buildable card is still safely buildable at time $i+1$. Since it was compliant, m_{i+1} must be the safe build of a card to foundation. Although we cannot delete m_i as we did in the first case, we can swap m_i and m_{i+1} to give the new sequence $m_{i+1}, m_i, m_{i+2} \ldots m_n$. The move m_i must remain legal as it did not involve the safely buildable card c_{i+1}. Move m_i may remain non-compliant but even if it does, it is one move nearer the end of the sequence. All other moves must remain legal and compliant.

The new sequence will either have fewer non-compliant moves than the original, or have the same number but with the last non-compliant move closer to the end of the sequence than before. This means that by repeated application of the process we will eventually obtain a sequence of moves that wins the game and has zero non-compliant moves. ■

It might seem that the proof would potentially be invalidated by worrying back cards from foundation to the tableau. However, the proof applies equally to them. Note that cards which are already in the foundation can still be potentially safely buildable, so they still count as possible cards that could be built in the tableau. Worrying-back a card that is potentially safely buildable is certainly pointless, so we can add the following simple corollary.

Corollary 2. *We can correctly add a dominance that disallows worrying-back a card from foundation if it would be immediately safely buildable after being worried back.*

Proof. We can enforce the dominance of moving safely buildable cards, which means the card must be immediately moved back to foundation without any non-foundation move intervening. Therefore the worry-back and rebuilding moves cancel out and can safely both be deleted. ■

Corollary 3. *The dominances we outlined in Section 5.4.1 for automatically building cards to foundation are correct. Specifically, for the following build policies a card is potentially safely buildable when the given condition holds:*

Red-black building with worrying back: *a card is at most two more than the top card on all foundations of the opposite colour and at most three more than the current card on foundation of the other suit of the same;*

Red-black building without worrying back: *either the previous condition holds, or the card is no more than one higher ranked than all the foundations of the opposite colour, or both;*

Building by suit: *a card is buildable to foundation, i.e. one higher than the highest card on the foundation of the same suit;*

Building regardless of suit: *a card is no more than two higher than the lowest card yet built to foundation of any suit.*

Proof. For each case we prove the condition is sufficient to ensure potential safe buildability by Definition 1.

Red-black building with worrying back:

For a card c of rank r which meets the conditions, then only a card d of rank $r-1$ of the opposite colour can be built onto c, but d is potentially buildable to foundation since each opposite colour foundation is of rank at least $r-2$. It would still be possible to build another card e of c's colour onto d: e must therefore be of rank $r-2$. But e is potentially buildable since all foundations of this colour are at least at rank $r-3$. Similarly d is potentially safely buildable since the only cards that can be moved to d are e or equivalent cards. All cards of lower rank than e are already on foundation: though they could be worried back to the tableau, they are themselves potentially safely buildable. Therefore c is safely buildable under this build policy.

Red-black building without worrying back:

The argument above for card c of rank r holds when that condition applies. For the additional condition, if all cards of opposite colour and rank $r-1$ are built to foundation, there are no cards which can possibly be built onto c in the tableau. In this case the second condition of being potentially safely buildable in Definition 1 is vacuous and trivially true.

Building by suit: Similar to the last case, if the card of rank $r-1$ and the same suit is on the foundation, then the second condition in Definition 1 is vacuous.

Building regardless of suit: For card c of rank r, since all cards of rank $r-2$ are on foundation, only cards of rank $r-1$ can be built on c in the tableau. But all these can be built to foundations themselves and are thus potentially safely-buildable.

■

C.2 Immediate Building After Tableau Moves

For an introduction to this dominance, see Section 5.4.2. To maximise the utility of proving correctness, we wish to generalise the dominance and also strengthen it slightly from its original form. The strengthening of the dominance is to insist that after a partial-pile move, the card above must be built immediately to foundation (not just be buildable in principle).

The generalisation over previous uses is to allow its application in cases which don't use a standard four-suit deck or the common red-black building policy. To do so we assume that the build policy has the property that, given any two cards, the two sets of places those cards can move to by the build policy are either identical or disjoint. We call a build policy an **indistinguishable** building policy if it both has no distinction between two cards which can move to the same place, and it has no distinction between the rule for building individual cards and for moving piles of cards in a block. For example, in the classic games using red-black building by rank, any two cards of different colours or ranks have disjoint cards they can be built on, while two cards of the same colour and rank can be built on exactly the same set of cards. Another indistinguishable policy would be in a game using five identical decks with three suits in which cards must be built in the same suit only: here there would be five copies of e.g. 9♠, each of which could be built on 10♠ but not on 10♣ or 10♡. Despite its flexibility this generalisation still excludes some build policies. An example is 'different-suit'. This does not meet the condition because both spades and diamonds can move to clubs, while spades but not diamonds can move to diamonds. Counterexamples to the theorem would occur if we allowed this build policy. For example, we might need a group headed by 9♠ to move from 10♣ to 10♢ to allow the 9♢ to move under the 10♣, as it cannot be moved to the 10♢. An indistinguishable build policy also requires that the same build policy controls moving groups and individual cards. Some games, such as Spider, use a policy where individual cards can be moved in any suit but built groups can only be moved if they are all the same suit. This means that moves of groups can be necessary to establish sequences of the same suit, even if the card above is not buildable.

Theorem 4. *We consider any patience or solitaire game which: has a tableau which builds down according to an 'indistinguishable' build policy (as defined above); allows moves of complete or incomplete built piles as a single move according to the same policy as for individual cards; the only place a card can move from the tableau is to another tableau pile or to a foundation; is won by moving all cards to the foundations; and contains no rules invalidating moves by constraints on their order in the move sequence. For any instance of such a game, if the instance is winnable with the original rules, then it is also winnable with the restriction that an incomplete built pile may only be moved if the card above the moved partial pile is then built immediately to foundation.*

Proof Consider any legal winning sequence of moves in the original rules, $m_1, m_2, \ldots, m_n$, containing at least one non-compliant move. We will create a new winning sequence of moves with either fewer non-compliant moves than the original, or have the same number but with the last non-compliant move closer to the end of the sequence than before.

Suppose that the move m_i is the last move in the sequence which is non-compliant, i.e. is the move of a partial pile not immediately followed by building the card above it to foundation. Note that $i < n$ since the last move in a winning game must be moving a card to the foundation pile. For m_{i+1} we do know: it exists since $i < n$; it is a legal move; it is not the move of the card of above it to foundation; and if m_{i+1} is a partial pile move then move m_{i+2} is building the card above it to foundation. We now show by case analysis how to replace moves m_i, m_{i+1} in the sequence. In most cases the adjustment is straightforward. Before describing the straightforward cases, we consider the most critical, difficult, case.

Case 1. The critical case is where move m_{i+1} is moving a card or pile onto the pile just vacated by the original move m_i (which was by hypothesis the last non-compliant move). We can illustrate by example in the case of a red-black build policy: this might be a move of a three card pile 10♣9♡8♠ from the J♢ to J♡, followed immediately by a move of the 10♠ to the J♢, i.e. $m_i = [10♣, J♡]$ and $m_{i+1} = [10♠, J♢]$. We deal with this case first by omitting move m_i, thus reducing the number of non-compliant moves by one, and then replacing m_{i+1} by a move $m'_{i+1} = [C(m_{i+1}), T(m_i)]$. In the example above, we would delete the move of 10♣ and change the move of the 10♠ to be to the J♡ instead of the J♢, i.e. $m'_{i+1} = [10♠, J♡]$. The move m'_{i+1} must be a legal move because the build policy is indistinguishable so cannot allow m_{i+1} and disallow m'_{i+1}. Move m'_{i+1} must also be compliant since move m_{i+1} was: i.e. if the move was of a partial pile then m_{i+2} must be building the card above to foundation.

We now have to consider the remaining moves in $m_{i+2}, \ldots m_n$. We create moves $m'_{i+2}, \ldots m'_j$ until we have identical layouts again in the original and new sequence of moves, after which we retain moves $m_{j+1} \ldots m_n$. Until then, we will maintain an invariant property, that the cards $T(m_i)$ and $T(m_{i+1})$ (J♢ and J♡ in our example) remain in the tableau, that at least one of them has a card built below it, that the piles under those cards are swapped in the new sequence compared to the original, and that all other cards in the layout are identical. This invariant certainly holds after the deletion of m_i and the replacement of m_{i+1} by m'_{i+1}. Now we assume the invariant is true up to move m'_{k-1} and consider move m_k. If this move does not involve either of the affected piles, then it necessarily retains the invariant, so we simply set $m'_k = m_k$. However, when the move does involve at least one of the affected piles, it must fall into one of the following five subcases. In the first three we have to adapt the sequence of moves to a new one to retain the invariant, with the last two being simpler.

(a) If move m_k is of $C(m_i)$ to $T(m_{i+1})$ (e.g. of 10♣ to J♢ in our example) then we simply delete the move completely. Because of the invariant the cards below $C(m_i)$ are already at the intended target location, so we need do nothing. Included in this sub-case is where move m_{i+1} is an exact reverse of m_i and both are deleted. In general, the final move sequence is

$$m_1, \ldots m_{i-1}, m'_{i+1}, \ldots m'_{k-1}, m_{k+1}, \ldots m_n$$

(b) If move m_k is of $T(m_{i+1})$ to foundation *[respectively $T(m_i)$]*, e.g. moves J♢ to foundation in our example *[respectively J ♡]*, then it now has a card built below it (by the invariant) so the move is not currently possible. By the invariant, the card $T(m_i)$ must itself be clear, since the card $T(m_{i+1})$ was clear in the original *[respectively $T(m_{i+1})$ must be clear]*. This means that we can now insert the move $m'_k = [C(m_i), T(m_i)]$ immediately before m_k *[respectively set $m'_k = [C(m_{i+1}), T(m_{i+1})]$]*. The move m'_k is legal by indistinguishability. It is a partial pile move where the immediately following move m_k will be of the card above it to foundation, so m'_k is a compliant move. The move m_k is now legal, positions are identical, and the new sequence contains one less non-compliant move. The final move sequence is

$$m_1, \ldots m_{i-1}, m'_{i+1}, \ldots m'_{k-1}, m'_k, m_k, m_{k+1}, \ldots m_n$$

(c) If move $m_k = [C_k, T_k]$ removes the last card below $T(m_i)$ and $T(m_{i+1})$, e.g. moves either 10♣ or 10♠ in our example, leaving both J♡ or J♢ uncovered, then we set m'_k to the same move $[C_k, T_k]$. The move m'_k must be legal, by indistinguishability. If the move is to foundation or the the move m_{k+1} is building the card above C_k to foundation, then m'_k is compliant and the sequence has one less non-compliant move. Even if m'_k is non-compliant, then the sequence has the same number of non-compliant moves as before (since we deleted m_i), but the last is nearer the end of the sequence (since $n - k > n - i$). By the invariant the layouts are now identical so the final move sequence is

$$m_1, \ldots m_{i-1}, m'_{i+1}, \ldots m'_{k-1}, m'_k, m_{k+1}, \ldots m_n$$

(d) If move m_k is from (or to) a card on a pile below either $T(m_i)$ or $T(m_{i+1})$, but is not covered by one of the above sub-cases (e.g. of 8♠ from the 9♡ to the 9♢ in our example) then we can make the unchanged move m_k now. That is, we move from $C(m_k)$ to $T(m_k)$: by the invariant the identical card that m_k was originally moved from (or to) is below the other one in the revised sequence. The invariant is thus retained.

(e) Any move m_k of a pile of cards starting from $T(m_i)$ or $T(m_{i+1})$ or any card above them can be retained unchanged with $m'_k = m_k$ and the invariant is retained.

All remaining cases are essentially straightforward because we can simply swap consecutive moves m_i and m_{i+1}, sometimes with minor changes. In each case the position after the second move is identical in each sequence, and we have either removed an non-compliant move or moved it one move closer to the end of the sequence, as required.

Case 2. If the moves m_i and m_{i+1} are entirely unrelated then we can simply swap the order of the moves as they do not affect each other. I.e. we create a new move sequence

$$m_1, \ldots m_{i-1}, m_{i+1}, m_i, m_{i+2}, \ldots m_n$$

Note that the swap cannot affect whether move m_{i+1} is compliant: by hypothesis it was a compliant move and it remains so. However, it is possible that, in its new position, the move m_i is now a compliant move. If that happens we have reduced the number of non-compliant moves, but if not we have moved the last non-compliant move one closer to the end of the move sequence.

Case 3. We can make consecutive moves *from* the same pile, i.e. have $C(m_{i+1})$ be either the card above $C(m_i)$ or a card in sequence above it. Because move m_i is non-compliant, the move m_{i+1} cannot be of the card above $C(m_i)$ to foundation, so the only remaining possibility is a second consecutive move between tableau piles. Again we can swap the order of the moves. The new move sequence is

$$m_1, \ldots m_{i-1}, m_{i+1}, m_i, m_{i+2}, \ldots m_n$$

Notice that in this case the card $C(m_i)$ (and any partial pile below it) is moved twice instead of just once, but ends in an identical position. The move m_{i+1} must still be compliant in the earlier position, while m'_i remains non-compliant, but appears one move nearer the end of the sequence.

Case 4. We can make consecutive moves *to* the same pile. In this case again we simply swap moves m_i and m_{i+1}. The analysis is the same as in the previous case, except that this time it is the card $C(m_{i+1})$ (and possibly partial pile below it) that is moved twice instead of once. Again, m_i remains non-compliant, but appears one move closer to the end of the sequence.

Case 5. The final possibility is that the second move is from the pile the first move went to. This gives a number of possibilities depending on the card moved the second time: the second card moved can be the same as the first card, a card above it in the new pile, or a card below it.

- If the same card is moved twice, then the moves cannot be an immediate reversal of moves since that was covered as case (a) in the Critical Case above. Thus, we can replace the two moves with a single move bypassing the intermediate position, $m_i' = [C(m_i), T(m_i + 1)]$. This move m_i' may still be non-compliant but is one move closer to the end. In this case the final sequence of moves is

$$m_1, \dots m_{i-1}, m_i', m_{i+2}, \dots m_n$$

- If the second card is either above or below the first moved card, then again we can just swap the order of the two moves, giving the sequence

$$m_1, \dots m_{i-1}, m_{i+1}, m_i, m_{i+2}, \dots m_n$$

  The result is the same, with the non-compliant move being one nearer the end of the sequence. If the card $C(m_{i+1})$ was above the first moved card in the second pile, then the card C_{m_i} and any pile below it is only are now only moved once instead of twice. If the card $C(m_{i+1})$ was below $C(m_i)$ then the card $C(m_i)$ and any cards below it are now moved only once.

In all of the cases analysed above, we are able to do one of two things. We either produce a new sequence with one less non-compliant move, or produce a sequence with the same number of non-compliant moves but the last one nearer the end of the sequence. Iterating this procedure must inevitably lead to a solution with zero non-compliant moves. Therefore we can impose the restriction without making any instance unsolvable. ■

We wish to use the two dominances together, and have to consider the possibility that they might be acceptable individually, but together make a winnable game unwinnable. Fortunately, it is straightforward to prove that this is not the case.

Theorem 5. *If the conditions of Theorem 1 and Theorem 4 both apply, then any winnable game has a winning sequence in which all moves are compliant with both dominances.*

Proof. If a game is winnable, by Theorem 4 it is also winnable while always building immediately after the move of an incomplete pile in the tableau. We can take this winning sequence as the starting point in the proof of Theorem 1. Each step of that proof retains winnability and moves towards a solution where the safe-building dominance is respected. The sequence changes in the proof of Theorem 1 concern some non-compliant move m_i which moves card c_i when some card was safely buildable. It is enough to check that no such change can produce a sequence which is non-compliant with the incomplete pile dominance. There were two cases, depending whether c_i was safely buildable or not at time i.

- If c_i was safely buildable, the proof of Theorem 1 delete move m_i and replaced it with the first move m_j building c_i to foundation, which occurred in a sequence of safe builds to foundation. The only affected move that could possibly have been of a partial pile is the first move m_i, which has now been deleted so the new sequence remains compliant with the incomplete pile dominance.
- If c_i was not safely buildable at time i, then the proof of Theorem 1 simply swapped m_i and m_{i+1}. But move m_i was compliant with the incomplete pile dominance, so if m_i was a partial pile move then m_{i+1} was the immediate build of the card above c_i to foundation. But this is an impossible combination because the earlier proof showed that the move c_i cannot have involved any safely buildable card and that m_{i+1} was the build of a safely buildable card.

■

In summary, we have proven the correctness of two key dominances, neither of which were previously proved correct. We have also shown that they can be used together if both are applicable.

Appendix D. Comparative Statistics With Alternative Solvers On Three Major Games

Our focus in this paper has been on obtaining winnability statistics using Solvitaire on the widest possible range of games. We have therefore not focussed on performance comparison of Solvitaire with existing solvers for games where good solvers exist. To do such a comparison to a high scientific standard to give meaningful results would itself require a major effort, even where the alternative solver is freely available. However, we have been able to run Solvitaire on identical instances with existing solvers for the three major games *Klondike*, *Canfield*, and *FreeCell*. These comparison gives a general indication of performance of the general purpose solver Solvitaire with solvers which were more specifically targeted at the relevant games.

The three solvers were: Jan Wolter's solver for *Canfield* (Wolter, 2014d); Matt Birrell's Klondike Solver for *Klondike* (Birrell, 2017); and Shlomi Fish's Freecell Solver (version 6.10) (Fish, 2024). For each game, both Solvitaire and the alternative solver were run on machines with identical specification (though the machines across different games were not identical). Timeouts varied between solvers: this was 30 seconds for *Canfield*, 5 minutes for *FreeCell*, and 1 hour for *Klondike*. Table 9 shows the results. The sample size of each experiment is given, together with how many instances each solver could determine correctly within the timeout. Remaining statistics only apply to those which could be determined correctly, and give statistics of time taken and nodes searched (where available). For *Canfield* and *Klondike* we had identified bugs in the original solvers as discussed in Section 7.1. For *Canfield*, we ran a minimally-corrected version of the solver, while for *Klondike* we discounted the 6 instances it reported incorrect results for.

For *Canfield*, we can see that within 5 minutes, the corrected version of Wolter's solver was able to solve about 2.2% more instances within 30 seconds, and also had lower run times in each statistic. So Solvitaire is not quite as good, but the performance penalty is relatively slight. In contrast, for *Klondike* the situation is reversed. Here, Solvitaire is able to solve more instances and many metrics of runtime are very much better. Finally, for *FreeCell* we report separate experiments on the first 10,000 seeds in Table 9 and (because of the rarity of unwinnable games) on the first 1,000 unwinnable seeds in Table 9. For the winnable games we report both Solvitaire used without streamliners, and with the 'smart' setting. It is noteable that Solvitaire without streamliners is very much worse than FC-Solve on winnable games, and still slightly worse on unwinnable games. For winnable games, the use of the smart streamliners is extremely effective and on some measures slightly outperforms FC-Solve. FC-Solve, however, does perform better that Solvitaire on unwinnable instances.

In summary, we can see that Solvitaire is able to perform well on each of these three classic games when compared to existing solvers for those games. For *Klondike* it is significantly better than the alternative, while it does not give as good performance as the alternatives for *Canfield* and *FreeCell*. It is also interesting to see the dramatic improvements given by the use of streamlining in *FreeCell*.

Full results of each solver on each instance are included in our online dataset (Gent & Blake, 2024).

Comparative results for **Canfield**. Time limit of 30s for each solver.

Algorithm	Sample	< limit	CPU time (seconds)					Kilonodes Searched				
			Mean	Median	90%	99%	Max	Mean	Median	90%	99%	Max
Wolter	50,000	48,979	0.6677	< 0.01	0.8900	16.41	29.98	219.3	0.216	276.5	5,449	13,650
Solvitaire	50,000	47,881	0.9543	0.020	1.71	19.99	29.95	79.06	0.551	141.5	1,658	4,318

Comparative results for **Klondike**. Time limit of 1hr for each solver.
Six instances were solved incorrectly by Klondike-solver.

Algorithm	Sample	< limit	CPU time (seconds)					Kilonodes Searched				
			Mean	Median	90%	99%	Max	Mean	Median	90%	99%	Max
Klondike-Solver	50,000	49,054	83.08	20.50	140.5	1433	3578	*not recorded*				
Solvitaire	50,000	49,656	32.45	0.020	14.93	905.7	3598	3,897	3.177	1,685	113,300	511,000

Results for **FreeCell** on the first 10,000 seeds (all winnable). Time limit of 5 minutes for each solver.
For Smart, initial run with streamliners reported one seed incorrectly but correctly found solution without streamliners.

Algorithm	Sample	< limit	CPU time (s)					Nodes Searched (thousands)				
			Mean	Median	90%	99%	Max	Mean	Median	90%	99%	Max
FC-Solve	10,000	9,998	0.1226	0.0500	0.0600	0.7503	53.99	28.59	0.302	8.096	300.3	19,460
Solvitaire (None)	10,000	9,746	6.839	0.1900	10.32	160.4	297.8	3,518	104.6	5,329	81,110	163,600
Solvitaire (Smart)	10,000	10,000	0.1496	0.0400	0.2800	1.870	23.24	75.1	19.93	144.3	891.0	10,960

Results for **FreeCell** on the first 1,000 unwinnable seeds. Time limit of 5 minutes for each solver.

Algorithm	Sample	< limit	CPU time (s)					Nodes Searched (thousands)				
			Mean	Median	90%	99%	Max	Mean	Median	90%	99%	Max
FC-Solve	1000	1000	0.3586	0.0800	0.2600	1.601	150.5	119.2	15.40	105.5	688.5	50,030
Solvitaire	1000	998	0.8135	0.1200	1.073	8.830	205.8	431.7	69.33	546.6	4,760	101,000

Table 9: Comparative results between Solvitaire and other solvers on *Canfield*, *Klondike* and *FreeCell*.

Appendix E. Summary Statistics of Experiments Reported Here

Table 10 summarises the results of Solvitaire on all patiences experimented on in this paper.

The first four columns report on the winnability statistics from which confidence intervals were calculated. The total sample is given as well as the number proven winnable, proven unwinnable, and unknown. In some cases, results for particular instances were not run on this particular game but deduced from related games, as described in Section 6.

The final four columns give summary search statistics for all our data provided in our auxiliary data. The number of runs is the number of times Solvitaire was executed on that specific game in our experimental set, and so therefore can be either higher or lower than the sample size in the first column. Lower numbers than the sample arises if other games are used to deduce results while higher numbers result from rerunning instances that Solvitaire initially failed to resolve. The final three columns give, to 2 significant figures, the mean number of nodes searched, the mean CPU time per run in CPU-seconds, and the total CPU time over all runs in CPU-days. These statistics are intended to give a general idea of the ease or difficulty that Solvitaire had with any game, as well as the total resources we devoted to that game. However, they are not well-suited for benchmarking Solvitaire against alternative solvers, because the focus of our experiments was to obtain high-quality estimates of winnability percentage. In particular, experiments were run on a variety of machines with different specifications, often varying within a single exeriment. CPU times are as recorded by our internal timing mechanism: while we did often record a slightly more accurate external timing mechanism, which was typically ≈ 10% higher, this statistic was not available for all instances.

With the exception of two games, full data for all instances we experimented on is provided in our online dataset (Gent & Blake, 2024). The exceptions are *British Canister* and *Fortune's Favor*, which were so easy and had winnability so close to 0/1 that we used samples of size 10^9. We only retained instances which are one of: in the first 10^7 instances; or took more than 1 sec. to solve; or had the rare result (winnable for *British Canister* or unwinnable for *Fortune's Favor*). While this means complete data is not available, it seemed a reasonable compromise between ability to check our work and excessive storage requirements. Note that, since all instances of both games were solved, the winnability of all 10^9 individual instances can be checked from our data.

Game	Variant	Winnability Statistics				Search Statistics			
		Sample	Winnable	Unwinnable	Unknown	Runs	Mean Nodes	Mean CPU (seconds)	Σ CPU (days)
Accordion		10^6	999,996	0	4	1,000,116	5.1×10^6	9.8	110
Alpha Star		10^7	4,779,474	5,220,526	0	10,000,000	7.7×10^2	0.0037	0.42
American Canister		10^7	560,567	9,439,428	5	10,000,179	9.1×10^4	0.61	71
Baker's Game		10^7	7,505,266	2,494,734	0	10,000,000	7.8×10^4	0.42	49
Beleaguered Castle		2×10^6	1,362,720	635,919	1,361	2,671,263	2.6×10^6	4.9	150
Black Hole		10^7	8,694,457	1,305,543	0	10,000,000	4.3×10^5	2.8	330

Game	Variant	Winnability Statistics				Search Statistics			
		Sample	Winnable	Unwinnable	Unknown	Runs	Mean Nodes	Mean CPU (seconds)	Σ CPU (days)
British Canister	†	10^9	1,290	999,998,710	0	*10,001,326*	*74*	*0.000095*	n/a
Canfield		10^7	7,124,239	2,875,241	520	10,000,000	1.6×10^6	4.8	560
Canfield	(Whole pile)	10^7	6,755,771	3,243,482	747	10,000,000	2.4×10^6	6.3	730
Delta Star		10^7	3,441,247	6,558,753	0	10,000,000	1.0×10^3	0.0035	0.4
East Haven		2×10^6	1,655,944	342,169	1,887	2,075,274	1.7×10^6	5.4	130
Eight Off		10^7	9,988,054	11,946	0	10,000,000	1.4×10^4	0.046	5.3
Fan		10^6	487,759	512,241	0	1,000,000	6.3×10^5	1	12
Fore Cell		10^7	8,561,569	1,438,082	349	10,000,000	3.6×10^5	0.71	82
Fore Cell	(BP =)	10^7	1,056,397	8,943,603	0	10,000,000	4.8×10^3	0.015	1.7
Fortunes Favor	†	10^9	999,999,881	119	0	*10,294,763*	*2.1×10^4*	*0.068*	n/a
FreeCell		10^7	9,999,890	110	0	10,000,016	7.6×10^4	0.37	43
FreeCell	(FC 0)	10^7	21,354	9,978,617	29	10,000,111	2.8×10^4	0.057	6.7
FreeCell	(FC 1)	10^6	193,335	806,370	295	1,000,749	1.4×10^6	3.3	39
FreeCell	(FC 2)	10^6	795,341	204,449	210	1,000,440	8.3×10^5	2.3	26
FreeCell	(FC 3)	10^6	993,580	6,410	10	1,000,021	1.4×10^5	0.32	3.7
FreeCell	(4 Piles)	10^7	864	9,999,136	0	10,000,000	1.5×10^3	0.0021	0.25
FreeCell	(5 Piles)	10^6	38,577	961,392	31	1,000,173	5.5×10^5	1.1	12
FreeCell	(6 Piles)	2×10^6	1,227,828	770,982	1,190	2,003,743	5.6×10^6	13	290
FreeCell	(7 Piles)	10^6	988,556	11,417	27	1,000,061	2.7×10^5	0.62	7.2
Gaps	(Basic Variant)	10^7	2,490,171	7,509,829	0	10,000,000	7.2×10^5	3.4	400
Gaps	(One Deal)	10^4	8,285	1,107	608	11,416	7.2×10^8	2,000	260
Golf		10^7	4,510,859	5,489,141	0	10,000,000	6.8×10^5	1.6	180
King Albert		2×10^6	1,370,321	628,618	1,061	2,011,590	4.8×10^6	15	360
Klondike		10^6	819,371	180,472	157	1,005,717	2.9×10^7	75	870
Klondike	(WB ×)	10^6	815,114	184,637	249	819,759	3.5×10^6	10	95
Klondike	(SP ✓, BP ∗)	10^6	999,233	763	4	3,366	1.1×10^7	26	1.0
Klondike	(SP ✓)	10^6	949,577	50,406	17	180,629	7.5×10^5	2.1	4.4
Klondike	(SP ✓,BP =)	10^6	407,620	592,380	0	931,055	1.4×10^4	0.032	0.34

Game	Variant	Winnability Statistics				Search Statistics			
		Sample	Winnable	Unwinnable	Unknown	Runs	Mean Nodes	Mean CPU (seconds)	Σ CPU (days)
Klondike	(BP $*$)	10^6	998,155	1,033	812	180,629	4.2×10^7	75	160
Klondike	(BP $=$)	10^6	68,945	931,055	0	1,000,000	4.0×10^4	0.063	0.73
Klondike	(SP $\times$, BP $*$)	10^6	24,068	1,134	974,798	2,645	2.8×10^8	600	18
Klondike	(SP $\times$)	10^6	20,757	977,411	1,832	819,528	3.9×10^7	99	940
Klondike	(SP $\times$,BP $=$)	10^6	1,772	998,228	0	68,945	3.3×10^4	0.051	0.040
Klondike	(Draw 1)	10^6	904,226	94,629	1,145	180,837	6.4×10^7	190	400
Klondike	(Draw 1,WB $\times$)	10^6	901,702	97,622	676	90,168	4.9×10^7	150	150
Klondike	(Draw 2)	10^6	885,476	113,084	1,440	511,234	2.3×10^7	60	360
Klondike	(Draw 2,WB $\times$)	10^6	882,409	116,624	967	167,750	4.4×10^7	110	220
Klondike	(Draw 4)	10^6	693,296	306,564	140	779,013	2.9×10^6	8.5	77
Klondike	(Draw 4,WB $\times$)	10^6	687,198	312,729	73	572,605	1.6×10^6	4.2	28
Klondike	(Draw 5)	10^6	534,329	465,656	15	999,774	1.0×10^6	3.3	39
Klondike	(Draw 5,WB $\times$)	10^6	526,376	473,621	3	494,742	3.4×10^5	0.94	5.4
Klondike	(Draw 6)	10^6	358,539	641,460	1	819,759	3.0×10^5	1	9.6
Klondike	(Draw 6,WB $\times$)	10^6	349,817	650,183	0	350,494	1.7×10^5	0.45	1.8
Klondike	(Draw 7)	10^6	237,786	762,214	0	1,000,000	1.6×10^5	0.45	5.2
Klondike	(Draw 7,WB $\times$)	10^6	229,522	770,478	0	237,755	1.2×10^5	0.31	0.86
Klondike	(Draw 8)	10^6	122,759	877,241	0	1,000,000	6.9×10^4	0.18	2.1
Klondike	(Draw 8,WB $\times$)	10^6	117,024	882,976	0	122,753	9.0×10^4	0.24	0.34
Klondike	(Draw 9)	10^6	76,699	923,301	0	819,759	5.1×10^4	0.13	1.3
Klondike	(Draw 9,WB $\times$)	10^6	72,140	927,860	0	76,676	7.6×10^4	0.2	0.18
Klondike	(Draw 10)	10^6	42,372	957,628	0	534,352	4.1×10^4	0.11	0.66
Klondike	(Draw 10,WB $\times$)	10^6	39,392	960,608	0	42,371	6.1×10^4	0.16	0.08
Klondike	(Draw 11)	10^6	20,655	979,345	0	905,431	1.3×10^4	0.034	0.36
Klondike	(Draw 11,WB $\times$)	10^6	19,037	980,963	0	20,654	4.5×10^4	0.12	0.029
Klondike	(Draw 12)	10^6	8,489	991,511	0	358,540	1.1×10^4	0.03	0.12
Klondike	(Draw 12,WB $\times$)	10^6	7,788	992,212	0	8,488	3.3×10^4	0.088	0.0087
Klondike	(Draw 13)	10^6	5,998	994,002	0	905,431	4.3×10^3	0.011	0.12

Game	Variant	Sample	Winnable	Unwinnable	Unknown	Runs	Mean Nodes	Mean CPU (seconds)	Σ CPU (days)
		Winnability Statistics				Search Statistics			
Klondike	(Draw 13,WB ×)	10^6	5,444	994,556	0	5,997	3.4×10^4	0.092	0.0064
Late-Binding Solitaire		10^7	4,702,154	5,297,846	0	10,000,000	6.7×10^4	0.054	6.3
Mrs Mop		2×10^6	1,958,661	38,969	2,370	2,004,805	1.3×10^7	38	880
Northwest Territory		10^6	683,669	316,287	44	1,001,297	4.9×10^6	21	240
Raglan		10^6	812,184	187,650	166	1,000,009	4.1×10^5	0.98	11
Seahaven Towers		10^7	8,933,178	1,066,822	0	10,000,000	8.4×10^4	0.12	14
Siegecraft		10^6	991,378	8,595	27	1,000,054	1.8×10^5	0.31	3.6
Simple Simon		10^6	974,476	25,467	57	1,000,000	6.0×10^4	0.15	1.7
Somerset		2×10^6	1,073,962	924,968	1,070	2,004,561	5.5×10^5	2.9	68
Spanish Patience		10^7	9,986,239	13,746	15	10,000,028	2.0×10^4	0.090	10
Spider		10^4	9,731	0	269	11,494	2.6×10^8	810	110
Spiderette		10^6	996,153	3,751	96	1,000,000	1.0×10^6	1.8	21
Streets and Alleys		2×10^6	1,021,425	973,933	4,642	2,012,134	1.6×10^7	29	670
Stronghold		10^6	973,689	26,106	205	1,000,320	1.7×10^6	3.4	40
Thirty		10^7	6,745,425	3,254,508	67	10,000,000	1.4×10^5	0.22	26
Thirtysix		10^6	946,196	52,704	1,100	1,001,085	9.2×10^6	18	200
Trigon		10^7	1,599,605	8,400,395	0	10,000,000	2.7×10^3	0.015	1.7
Will o' the Wisp		10^7	9,992,300	7,487	213	10,000,906	2.9×10^5	1.3	150
Worm Hole		10^6	998,881	1,104	15	1,000,662	2.3×10^7	41	470

Table 10: Summary statistics for winnability and search for experiments reported in this paper. For search statistics, number of runs is precise with other figures given to two significant figures. † Run times for the full set of 10^9 instances were not recorded - see main text on page 48. WB : Worrying back. SP : spaces. BP : Build Policy, FC: number of free cells.

Appendix F. Rule Description Language

Our rule description language is defined as a JSON schema (Droettboom, 2023, Draft 4). For convenience to the user in specifying games, each parameter has a default value which is used if it is not explicitly overridden. The default values for every parameter are shown in Listing 4: they define the existing game *Streets and Alleys*. The JSON schema we use to parse and validate a user-defined ruleset is shown in Listing 5. As JSON schemas are not able to express every condition defining a valid game, Solvitaire also has a secondary post-schema validation step in code, outlined in the comments below. We cannot guarantee that every expressible game under this schema is handled correctly due to the large number of rule combinations, though many variants have been tested.

Listing 4: Rules of *Streets and Alleys* in our JSON format. These are also default values used for any game where that value is unspecified. The fields 'accordion' and 'sequences' are used for *Accordion*-like and *Gaps*-like games respectively.

```
"tableau piles": {
    "count": 8,
    "build policy": "any-suit",
    "spaces policy": "any",
    "diagonal deal": false,
    "move built group": "no",
    "move built group policy": "same-as-build",
    "face up cards": "all" },
"foundations": {
    "present": true,
    "initial cards": "none",
    "base card": "A",
    "removable": false,
    "only complete pile moves": false },
"hole": {
    "present": false,
    "base card": "AS",
    "build loops": true },
"cells": {
    "count": 0
    "pre-filled": 0 },
"stock": {
    "size": 0,
    "deal type": "waste",
    "deal count": 1,
    "redeal": false },
"reserve": {
    "size": 0,
    "stacked": false },
"accordion": {
    "size": 0,
    "moves": [],
    "build policies": [] },
"sequences": {
    "count": 0,
    "direction": "L",
    "build policy": "same-suit",
    "fixed suit": false },
"max rank": 13,
"two decks": false
```

Listing 5: A JSON schema defining the rule description language for games in Solvitaire. Comments specify additional constraints not covered by the schema itself.

```
"$schema": "http://json-schema.org/draft-04/schema#",
"description": "JSON Schema representing a generic solitaire game",
"type": "object",
"properties": {
  "tableau piles": {
    "type": "object",
    "properties": {
      "count": {
        "type": "integer",
        "minimum": 0},  // must be < deck size (= 4 * ["max rank"] (* 2 if ["two decks"]))
      "build policy": {
        "type": "string",
        "enum": [
          "any-suit",
          "red-black",
          "same-suit",
          "no-build"]},
      "spaces policy": {
        "type": "string",
        "enum": [
          "any",
          "no-build",
          "kings",  // ["max rank"] must be 13
          "auto-reserve-then-any",  // ["reserve"]["size"] must be > 0
          "auto-waste-then-stock",  // ["stock"]["size"] > 0 and ["stock"]["deal type"] is "waste"
          "auto-reserve-then-waste"]},  // both of the above conditions
      "diagonal deal": {
        "type": "boolean"},
      "move built group": {
        "type": "string",
        "enum": [
          "yes",
          "no",  // ["move built group policy"] ignored
          "whole-pile",
          "maximal-group",
          "partial-if-card-above-buildable"]},
      "move built group policy": {
        "type": "string",
        "enum": [
          "same-as-build",
          "any-suit",
          "red-black",
          "same-suit",
          "no-build"]},
      "face up cards": {
        "type": "string",
        "enum": [
          "all",
          "top"]}},
    "additionalProperties": false},
  // one and only one of [foundations], [hole], [accordion][size] > 0 and [sequences][count] > 0
  // must be present
  "foundations": {
    "type": "object",
    "properties": {
      "present": {
        "type": "boolean"},
```

```
    "initial cards": {
      "type": "string",
      "enum": [
        "none",
        "one",
        "all"]},
    "base card": {
      "type": "string",
      "oneOf":[
            {"pattern": "^(([0-9]|1[0-3]|a|A|j|J|q|Q|k|K))$"},  // must respect ["max rank"]
            {"enum": ["random"]}]},
    "removable": {
      "type": "boolean"},
    "only complete pile moves": {
      "type": "boolean"}},
  "additionalProperties": false},
"hole": {
  "type": "object",
  "properties": {
    "present": {
      "type": "boolean"},
    "base card": {
      "type": "string",
      "oneOf":[
            {"pattern": "^(([0-9]|1[0-3]|a|A|j|J|q|Q|k|K)(c|C|d|D|s|S|h|H))$"},
            {"enum": ["random"]}]},
    "build loops": {
      "type": "boolean"}}},
"cells": {
  "type": "object",
  "properties": {
    "count": {
      "type": "integer",
      "minimum": 0},
    "pre-filled": {  // must be less than deck size
      "type": "integer",
      "minimum": 0}},
  "additionalProperties": false},
"stock": {
  "type": "object",
  "properties": {
    "size": {  // must be less than deck size
      "type": "integer",
      "minimum": 0},
    "deal type": {
      "type": "string",
      "enum": [  // waste / tableau / hole must be specified
        "waste",
        "tableau piles",
        "hole"]},
    "deal count": {  // // must be less than ["stock"]["size"]
      "type": "integer",
      "minimum": 1},
    "redeal": {
      "type": "boolean"}},
  "additionalProperties": false},
"reserve": {
  "type": "object",
  "properties": {
    "size": {
      "type": "integer",
```

```
        "minimum": 0},  // must be less than deck size
      "stacked": {
        "type": "boolean"}},
    "additionalProperties": false},
  "accordion": {
    "type": "object",
    "properties": {
      "size": {  // must be less than deck size
        "type": "integer",
        "minimum": 0},
      "moves": {
        "items": {
          "type": "string",
          "pattern": "^((L|R)([1-9]|[1-4][0-9]|5[0-2]))$"}},
      "build policies": {
        "type": "array",
        "items": {
          "type": "string",
          "enum": [
            "same-suit",
            "red-black",
            "any-suit",
            "same-rank"]}}}},
    "additionalProperties": false},
  "sequences": {
    "type": "object",
    "properties": {
      "count": {  // must be less than deck size
        "type": "integer",
        "minimum": 0},
      "direction": {
        "type": "string",
        "enum": [
          "L",
          "R",
          "LR"]},
      "fixed suit": {
        "type": "boolean"},
      "build policy": {
        "type": "string",
        "enum": [
          "any-suit",
          "red-black",
          "same-suit"]}},
    "additionalProperties": false},
  "max rank": {
    "type": "integer",
    "minimum": 1,
    "maximum": 13},
  "two decks": {
    "type": "boolean"}},
"additionalProperties": false
```