Monotone Grid Classes: Limit Shapes and Enumeration

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The (monotone) *grid class* $\mathsf{Grid}(M)$ is a permutation class defined by a *gridding matrix* M whose entries are drawn from $\{$⧄, ⧅, □$\}$. This matrix specifies the permitted shape for plots of permutations in the class. Each entry of M corresponds to a *cell* in an *M-gridding* of a permutation. If the entry is ⧄, any points in the cell must form an increasing sequence; if the entry is ⧅, any points in the cell must form a decreasing sequence; if the entry is □, then the cell must be empty. With a minor abuse of terminology, we often refer to the matrix entries themselves as cells, calling □ entries *blank* cells, and ⧄ and ⧅ entries *non-blank* cells. A permutation can have more than one M-gridding. See Figure 1 for an example.

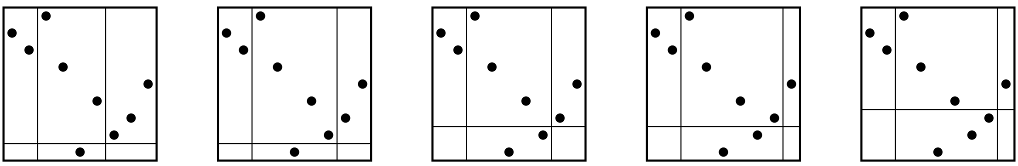
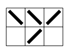


Figure 1: The five -griddings of 879614235

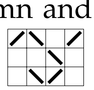
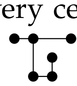
In this talk, we will consider the following questions:

- What does a typical large permutation in a given monotone grid class look like? That is, what is the *limit shape* of the class?
- What is known (and what is not known) about the enumeration (exact and asymptotic) of monotone grid classes?

The presentation will include several open questions, of varying perceived difficulty.

The Distribution of Points between Cells

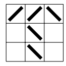
The *gridded class*, consisting of all M-gridded permutations, is denoted by $\mathsf{Grid}^\#(M)$. If $\sigma^\# \in \mathsf{Grid}^\#(M)$ then $\sigma^\#_{(i,j)}$ denotes the number of points of $\sigma^\#$ in cell (i,j).

A grid class $\mathsf{Grid}(M)$ is *connected* if it has a connected *cell graph*, which is the graph whose vertices are the non-blank cells of M, and in which two vertices are adjacent if they share a row or column and every cell between them is blank. For example, the (connected) cell graph of  is .

Key to our results is determining the proportion of points that occur in each cell in a typical large M-gridded permutation. If $\mathsf{Grid}(M)$ is connected, then there is a unique, explicitly computable, real *M-distribution* matrix $\Gamma_M = (\gamma_{i,j})$ such that for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\big[\max_{i,j} \big|\boldsymbol{\sigma}^\#_{(i,j)}/n - \gamma_{i,j}\big| \leqslant \varepsilon\big] \;=\; 1,$$

where, for each n, the gridded permutation $\boldsymbol{\sigma}^\#$ is drawn uniformly from $\mathsf{Grid}^\#_n(M)$.

For example, if $M_{\mathsf{T}} =$ , then $\Gamma_{M_{\mathsf{T}}} = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & 0 \end{pmatrix}$.

Limit Shapes

Given a permutation class, it may be that almost all large permutations in the class have the "same shape". To formalise this idea, we make use of certain probability measures which act as analytic limits of sequences of permutations. A *permuton* is a probability measure μ on the unit square $[0,1]^2$ with uniform marginals. That is, $\mu([a,b] \times [0,1]) = \mu([0,1] \times [a,b]) = b - a$, for every $0 \leqslant a \leqslant b \leqslant 1$. There is a natural topology on the space of permutons obtained by restricting the weak topology on probability measures.

A permuton μ_σ can be associated with an n-permutation σ by taking its plot, scaling it into the unit square, and replacing each point with a square of area $1/n^2$ and density n, as illustrated in Figure 2.

Figure 2: A plot of the permutation $\pi = 314592687$, and a picture of the permuton μ_π

Given a permutation class $\mathcal{C}$, let $\boldsymbol{\sigma}_n$ be an n-permutation drawn uniformly at random from $\mathcal{C}_n$. If the sequence of random permutons $(\mu_{\boldsymbol{\sigma}_n})_{n \geqslant 1}$ converges in distribution for the weak topology to some (possibly random) permuton $\boldsymbol{\mu}$, then we say that $\boldsymbol{\mu}$ is the *limit shape* of $\mathcal{C}$.

Every connected grid class has a *deterministic* limit shape whose support consists of oblique line segments. For example, the limit shape for $\mathsf{Grid}(M_{\mathsf{T}})$ is shown in Figure 3, the top row and central column both having width $\frac{2}{3}$.

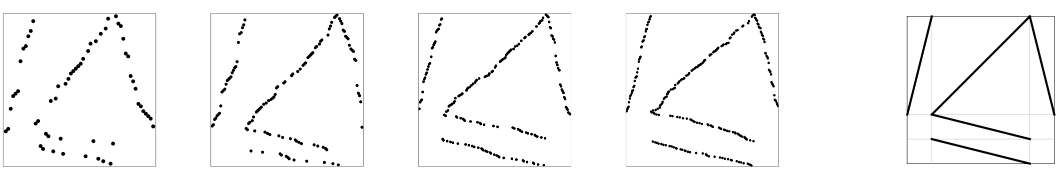


Figure 3: Plots of permutations of length 60, 120, 180 and 240 in $\mathsf{Grid}(M_{\mathsf{T}})$, and the limit shape of the class

Enumeration

Any monotone grid class whose cell graph is *acyclic* has a rational generating function. However, this result is nonconstructive, and no effective procedure is known to establish the generating function of an arbitrary acyclic class. On the other hand, acyclic *gridded* classes also have rational generating functions, which are straightforward to compute. This provides us with a strategy for determining the *asymptotic* enumeration of a connected acyclic grid class $\mathsf{Grid}(M)$:

1. Find Γ_M, as described above.
2. For each $\ell \geqslant 1$, determine how a typical large M-gridded permutation $\sigma^\#$ must be structured so that its underlying permutation σ has ℓ distinct M-griddings. (Rare atypical structures can be ignored.) This involves analysing how the row and column dividers may move, as in Figure 1.
3. Let $\sigma_n^\#$ be a gridded permutation drawn uniformly at random from $\mathsf{Grid}_n^\#(M)$, and let σ_n be its underlying permutation. By combining steps 1 and 2, calculate, for each $\ell \geqslant 1$, the asymptotic probability
$$P_\ell \;:=\; \lim_{n\to\infty} \mathbb{P}\big[\sigma_n \text{ has exactly } \ell \text{ distinct } M\text{-griddings}\big].$$
4. From its generating function, determine the asymptotic enumeration of the corresponding gridded class: $\big|\mathsf{Grid}_n^\#(M)\big| \sim \theta^\# g^n$, where g is the exponential growth rate of the class, and $\theta^\#$ is a constant.
5. Let $\kappa_M := \sum_{\ell\geqslant 1} P_\ell/\ell = \lim_{n\to\infty} \dfrac{\big|\mathsf{Grid}_n(M)\big|}{\big|\mathsf{Grid}_n^\#(M)\big|}$. Then $\big|\mathsf{Grid}_n(M)\big| \sim \kappa_M\, \theta^\#\, g^n$.

For example, we have the following for $\mathsf{Grid}(M_\mathsf{T})$:

1. Γ_{M_T} is given at the top of the previous page.
2. Depending on the position of points relative to the row and column dividers, a typical large M_T-gridded permutation $\sigma^\#$ may be such that its underlying permutation σ has either 1, 2 or 4 distinct griddings.
3. The asymptotic probabilities are $P_1 = \frac{1}{9}$, $P_2 = \frac{4}{9}$ and $P_4 = \frac{4}{9}$.
4. The gridded class has generating function
$$F_{M_\mathsf{T}}^\#(z) \;:=\; \sum_{n\geqslant 0} \big|\mathsf{Grid}_n^\#(M_\mathsf{T})\big|\, z^n \;=\; \frac{1}{1-5z+4z^2}.$$
Thus, $\big|\mathsf{Grid}_n^\#(M_\mathsf{T})\big| \sim \frac{4}{3} \times 4^n$.
5. The correction factor $\kappa_{M_\mathsf{T}} = \frac{4}{9}$. Therefore, $|\mathsf{Grid}_n(M_\mathsf{T})| \sim \frac{16}{27} \times 4^n$.

References

[1] Michael Albert, M. D. Atkinson, Mathilde Bouvel, Nik Ruškuc, and Vincent Vatter (2013). Geometric grid classes of permutations. *Trans. Amer. Math. Soc.*, 365(11):5859–5881.

[2] Michael Albert and Vincent Vatter (2019). An elementary proof of Bevan's theorem on the growth of grid classes of permutations. *Proc. Edinburgh Math. Soc.*, 62(4):975–984.

[3] Noura Alshammari and David Bevan (2024). On the asymptotic enumeration and limit shapes of monotone grid classes of permutations. https://arxiv.org/pdf/2408.00607

[4] David Bevan (2015). Growth rates of permutation grid classes, tours on graphs, and the spectral radius. *Trans. Amer. Math. Soc.*, 367(8):5863–5889.

[5] David Bevan (2015). *On the growth of permutation classes*. PhD thesis, The Open University.