Pattern packings, linear extensions, and moment sequences

Natasha Blitvić Queen Mary University of London

This talk is based on joint work with Slim Kammoun and Einar Steingrímsson

Definition 1. Fix $k, j \in \mathbb{N}$, $\pi \in S_k$, and $m \in \{1, \dots, k\}$. We say that a permutation σ is *m-packing j occurrences of the consecutive permutation pattern* π if σ contains occurrences of π starting in positions $\{1, 1+k-m, 1+2(k-m), \dots, 1+(j-1)(k-m)\}$.

In other words, given a fixed consecutive pattern $\pi \in S_k$, we consider permutations $\sigma \in S_n$ for $n \geq k$ (in fact, $n = k + (j-1)(k-m)$) in which π is guaranteed to occur at regular intervals. (Note that Definition 1 does not preclude additional occurrences, outside of the regularly-occurring ones.) For example, the permutation 14263758 is 2-packing 3 occurrences of 1324: namely, 1426, 2637 and 3758.

At first sight, the definition may seem familiar. Without explicitly keeping track of the positions of the occurrences, and instead considering permutations $\sigma \in S_n$ such that every position $i \in \{1, \dots, n\}$ is included in some occurrence of π, one can interpret σ as a 'cluster', in the sense of the cluster method of Goulden and Jackson. (See e.g. [1, 12] for some enumerative consequences of this point of view.)

Moving beyond clusters and tracking the *positions* of occurrences of a consecutive pattern is also natural: it generalizes the notion of the descent set of a permutation. Special cases of the construction of Definition 1 have appeared in the combinatorics literature [7, 8, 9, 10], but, to the best of our knowledge, these objects have never been considered in the generality presented here. Furthermore, drawing a permutation σ at random, the starting positions occurrences of a given pattern π gives rise to a *point process* (a natural generalization of the descent-tracking point process in [2]). When the permutation is drawn uniformly at random, describing the point process, in terms of its *correlation functions*, reduces to counting permutations σ containing occurrences of π in prescribed positions. (And when the permutation is not uniform, but is drawn according to some conjugacy-invariant law, we show that the conclusion is very similar – we shall touch upon this briefly in our lecture as well.)

More to the point, in addition to distinguishing the positions of the occurrences of π, we also require these to occur at regular intervals, with prescribed 'overlaps' m. This regularity is key. In particular, the inequalities satisfied by the elements of any such permutation σ give rise to posets (not to be confused with the posets in [11]) that themselves exhibit regularity. For a given pattern π, the number of permutations m-packing j occurrences of π equals the number of linear extensions of a poset defined in terms of π, m and j. An element in such a poset can cover (and be covered by) at most two elements, and there are further limitations on their structure.

Taken together, these definitions lead us to the following two classes of results, which we will discuss in this talk:

- The posets arising from packings of consecutive permutation patterns give rise to interesting classification problems. Furthermore, the poset point of view, combined with a probabilistic perspective, opens up a broader class of enumerative techniques and yields **new enumerative results** for permutations packing consecutive patterns. These take the form of explicit integer sequences, generating functions, integrals, or matrix products, and extend well beyond the initial exploration in [3].

- The regularity of the construction, for reasons that will be discussed in the lecture, suggests a link to moment sequences. Acting on an intuition originally presented in [3], we introduce **new general families of combinatorial moment sequences** associated with consecutive permutation patterns – and conjecture many more. In particular, Definition 1 provides a setting in which analogues of the recent 'positivity' questions associated with permutation patterns (see e.g. [4, 5, 6]) can be systematically studied and answered for *consecutive* permutation patterns.

References

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