Avoiding a Single Pattern in the Bruhat Order

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The Bruhat order on $\mathfrak{S}_n$ is a classical partial order that naturally arises in the study of $\mathfrak{S}_n$ as a Coxeter group. In particular pattern avoidance naturally arises in classifying elements whose principal order ideals are isomorphic to subsets under inclusion as was shown by Tenner in [5] . Alternatively it was shown by Barcucci, Bernini, Ferrari, and Poneti in [1] that when considering $Av_n(312)$ as a subposet of the Bruhat order the resulting partial order is the distributive lattice corresponding to Dyck Paths partially ordered by inclusion.

We attempt to study the subposets of the Bruhat order corresponding to $Av_n(\sigma)$ for general $\sigma \in \mathfrak{S}_k$. In particular we focus on the case where $\sigma = 2143$. Our main results concerning these posets can be summarized as follows.

Proposition 1. *If $\sigma \notin \{1, 12, 21, 132, 213, 231, 312\}$ then $Av_n(\sigma)$ is not a lattice.*

Theorem 2. *There is an order preserving map $\Phi : Av_n(2143) \to Av_n(132)$ that preserves the number of inversions satisfying $\Phi^2 = \Phi$.*

A generalization of permutations are *alternating sign matrices*.

Definition 3 (Alternating Sign Matrices)**.** An *alternating sign matrix* is a square matrix where each entry is either $0, 1$, or -1, every row and column sum is 1, and the non-zero entries in each row alternate in sign between 1 and -1. Denote the set of all $n \times n$ alternating sign matrices by ASM_n.

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix} \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix} \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix} \begin{bmatrix}0&1&0\\1&-1&1\\0&1&0\end{bmatrix} \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix} \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$$

Figure 1: The alternating sign matrices of ASM_3

One way of realizing this partial order is via the *corner sum matrix* of $\pi \in \mathfrak{S}_n$. The alternating sign matrices arise naturally in the study of the Bruhat order. Ihey also can be partially ordered via the corner sum matrix order and this order is the lattice completion of the Bruhat order.

Definition 4. Let $A \in ASM_n$. The *corner sum matrix of σ*, denoted by $csm(\sigma)$, is the $n \times n$ matrix where $csm(i, j) = \sum_{l \leq i, k \leq j} a_{l,k}$. The partial order on alternating sign matrices is the partial order defined by $A \geq B$ if for all $1 \leq i, j \leq n$, $csm(A)_{i,j} \leq csm(B)_{i,j}$.

In particular there is a notion of what it means for an alternating sign matrix to contain a pattern.

Definition 5 ([4])**.** An alternating sign matrix A contains an occurrence of a permutation π if there is a submatrix D where $D_{ij} = 1$ if $\pi(i) = j$. Denote by $ASMAv_n(\pi)$ the set of $n \times n$ alternating sign matrices which do not contain an occurrence of π.

$$\begin{bmatrix}1&1&1\\1&2&2\\1&2&3\end{bmatrix}\begin{bmatrix}1&1&1\\1&1&2\\1&2&0\end{bmatrix}\begin{bmatrix}0&1&1\\1&2&2\\1&2&3\end{bmatrix}\begin{bmatrix}0&0&1\\1&1&2\\1&2&3\end{bmatrix}\begin{bmatrix}0&1&1\\1&1&2\\1&2&3\end{bmatrix}\begin{bmatrix}0&1&1\\0&1&2\\1&2&3\end{bmatrix}\begin{bmatrix}0&0&1\\0&1&2\\1&2&3\end{bmatrix}$$

Figure 2: The corner sum matrices of ASM_3

$$\begin{bmatrix}0&0&0&0&1&0\\0&0&0&1&-1&1\\0&\color{red}{1}&0&-1&1&0\\0&0&1&0&0&0\\1&0&-1&\color{red}{1}&0&0\\0&0&\color{red}{1}&0&0&0\end{bmatrix}$$

Figure 3: A 6×6 alternating sign matrix with an occurrence of 132 highlighted.

In recent years there has been a surge of activity concerning alternating sign matrices avoiding a permutation pattern [2, 3], which was first studied in [4]. However understanding the partial order structure of alternating sign matrices avoiding a pattern does not seem to be studied. We show that the restriction of the Bruhat order to $Av_n(312)$ of [1] can be generalized to the setting of alternating sign matrices avoiding a single nonmonotonic pattern of length 3 as follows.

Theorem 6. *The corner sum matrix partial order on alternating sign matrices restricted to $AvASM_n(132)$ is isomorphic to the partial order on Schröder paths ordered by reverse containment.*

References

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