Permutations with only reduced co-BPDs

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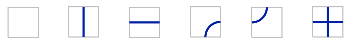
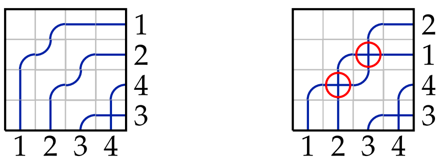
This talk is based on joint work with Adam Gregory

We give a pattern-avoidance characterization of a certain set of permutations arising from Schubert calculus, solving a problem of Weigandt [9]. We begin by introducing the definitions needed to state our result. Afterwards, we provide context and discuss our motivation for finding such a characterization.

A *bumpless pipe dream* (BPD) of size n is a tiling of an $n \times n$ grid with tiles of the form



such that n pipes enter from the bottom and exit to the right [3]. Labeling the pipes in increasing order from left-to-right as the enter, one obtains an *associated permutation* by reading the pipe labels from top-to-bottom as they exit, treating multiple crossings between a pair of pipes as turns if such an occurrence exists. A BPD is *reduced* if no two pipes cross more than once and *non-reduced* otherwise.

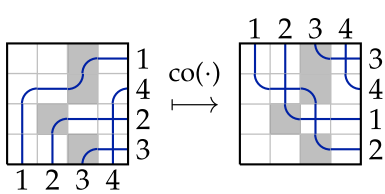


reduced non-reduced

Write BPD(w) for the bumpless pipe dreams with associated permutation w. We refer to the and tiles as *elbows*. Every BPD is completely determined by the location of its elbows [8]. Note that BPDs are in bijection with alternating sign matrices (ASM) by replacing each with a 1, each with a -1, and all other tiles with a 0 [8]. In recent work, Weigandt defined a co-BPD object corresponding to each BPD.

Definition 1 ([9])**.** For a given bumpless pipe dream B, its corresponding *co-bumpless pipe dream* co(B) is defined by exchanging tiles as follows:

In other words, the locations of the elbows are the same, but now pipes enter from the top instead of the bottom while still exiting to the right. Labeling pipes in increasing order from left-to-right as they enter, co(B) *traces out* a permutation by reading the pipe labels from top-to-bottom as they exit, treating any multiple crossings as turns. The permutation associated to B may be different from the one co(B) traces out.



A co-BPD is *reduced* if no two pipes cross more than once and *non-reduced* otherwise. Notice that $\mathrm{co}(\cdot)$ does not necessarily preserve reducedness as seen in the example above. We refer to the set $\{\mathrm{co}(B) : B \in \mathrm{BPD}(w)\}$ as the co-BPDs for w.

We now state our main result.

Theorem 2. *All co-BPDs for a permutation w are reduced if and only if w avoids the seven patterns 1423, 12543, 13254, 25143, 215643, 216543, and 241653.*

Motivation

We now provide context to motivate our result.

Schubert polynomials $\mathfrak{S}_w$ are a celebrated family of functions indexed by permutations $w \in S_n$. They generalize Schur functions and represent cohomology classes of Schubert varieties in the complete flag variety [4]. Grothendieck polynomials $\mathfrak{G}_w$ are K-theoretic generalizations of Schubert polynomials [5].

There are several known combinatorial formulas for $\mathfrak{S}_w$. One of the earliest is the *pipe dream* (PD) model of Billey–Bergeron [1] that extends to $\mathfrak{G}_w$ by Fomin–Kirillov [2]. A more recent one is the *bumpless pipe dream* (BPD) model of Lam–Lee–Shimozono [3] that was also extended to $\mathfrak{G}_w$ by Weigandt [8] using the connection to ASMs.

For a permutation w, let $\mathrm{bpd}(w) \subseteq \mathrm{BPD}(w)$ be the subset of bumpless pipe dreams for w that are reduced. For $B \in \mathrm{BPD}(w)$, define $\mathrm{blank}(B)$ to be the set of locations of each tile and $\mathrm{jay}(B)$ to be the set of locations of each tile. We can then define $\mathfrak{S}_w$ and $\mathfrak{G}_w$ as sums over bumpless pipe dreams given the following monomial weighting:

$$\mathfrak{S}_w = \sum_{B \in \mathrm{bpd}(w)} \prod_{(i,j) \in \mathrm{blank}(B)} x_i \ ; \qquad ([3])$$

$$\mathfrak{G}_w = \sum_{B \in \mathrm{BPD}(w)} (-1)^{\ell(w)} \left(\prod_{(i,j) \in \mathrm{blank}(B)} -x_i \right) \left(\prod_{(i,j) \in \mathrm{jay}(B)} 1 - x_i \right) . \qquad ([8])$$

Both families of polynomials are bases for the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$. Their change of bases formulas have been studied combinatorially by both Lenart [7] and Lascoux [6] using the pipe dream model. Weigandt introduced these co-BPD objects to provide analogous formulas using bumpless pipe dreams.

Theorem 3 ([9])**.** *Let $a_{w,v} = \#\{B \in \mathrm{BPD}(w) : \mathrm{co}(B)$ reduced & traces out $v\}$. Then*

$$\mathfrak{G}_w = \sum_v (-1)^{\ell(v)-\ell(w)} a_{w,v} \cdot \mathfrak{S}_v \ .$$

Theorem 4 ([9])**.** *Let $b_{w,v} = \#\{B \in \mathrm{bpd}(w) : \mathrm{co}(B)$ traces out $v\}$. Then*

$$\mathfrak{S}_w = \sum_v b_{w,v} \cdot \mathfrak{G}_v \ .$$

Theorem 3 naturally leads to asking which permutations have $\sum_v a_{w,v} = |\mathrm{BPD}(w)|$, which is answered by our main result Theorem 2. It is interesting to note that the solution in the BPD setting is more complicated than the corresponding one in the pipe dream setting, where it is simply 132-avoiding (i.e., dominant) permutations.

Future Directions

For a given w, little is known about the possible v that can contribute to $a_{w,v}$. However, as a corollary of our main result we can at least say the following.

Corollary 5. *If B is a non-reduced BPD, then $\mathrm{co}(B)$ traces a permutation containing the reverse of at least one of the patterns from Theorem 2.*

On the other hand, we wonder what one can say about the set of permutations *containing* one of the patterns 1423, 12543, 13254, 25143, 215643, 216543, or 241653 from Theorem 2. Notably, is there an alternate description of this set of permutations using other notions of pattern containment?

References

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