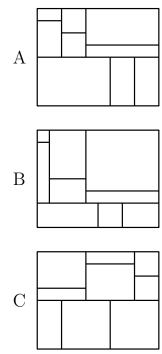
$\top$-AVOIDING RECTANGULATIONS AND $I(010,101,120,201) \cong I(011,201)$

Andrei Asinowski[1] University of Klagenfurt

This talk is based on joint work with Michaela Polley [2] *(Dartmouth College).*

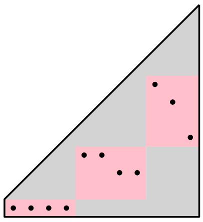
Abstract. We show that $\top$-avoiding rectangulations are in bijection with several classes of inversion sequences, among them $I(010,101,120,201)$ and $I(011,201)$. This provides a proof for the conjecture [7] that $I(011,201)$ is enumerated by A279555, and finalizes the classification of classes of inversion sequences, determined by four patterns of length 3, which are enumerated by this sequence [3].

Rectangulations. A *rectangulation* of size n is a partition of a rectangle R into n rectangles. Rectangulations form a combinatorial class under either of two kinds of equivalence: *weak equivalence* which preserves segment-to-rectangle adjacencies, and *strong equivalence* which (also) preserves rectangle-to-rectangle adjacencies. In the image, all three rectangulations are weakly equivalent, but only A and B are strongly equivalent. Equivalence classes are, accordingly, referred to as *weak rectangulations* and *strong rectangulations*. See [1] for more details, summary of basic results, and recent developments.



Patterns in Rectangulations implicitly appeared in several early contributions, but systematic investigation only started recently; see for example [5]. In [2], we presented enumerative results for rectangulations that avoid the pattern $\top$ and its rotations, in all possible combinations. In the present contribution, we further investigate $\top$-avoiding strong rectanulations — which, surprisingly, leads to a solution of a conjecture concerning enumeration of avoidance classes of inversion sequences.

Inversion sequences enumerated by OEIS A279555. The sequence A279555 was contributed by Martinez and Savage as the enumerating sequence of two classes of inversion sequences, $I(010,110,120,210)$ and $I(010,100,120,210)$ [4]. We also consider the avoidance class $I(010,101,120,201)$. It is very easy to show that it is equinumerous with both former classes, but it has a simpler structure as shown in the image: all the points of the plot of such inversion sequences are contained in an ascending sequence of rectangular areas (in pink), the points within every rectangular area are weakly descending, and one of them lies in the top-left corner of the area.



In [7, Conjecture 8.3], Yan and Lin conjectured that $I(011,201)$ and $(011,210)$ are also enumerated by A279555. Pantone [6, Section 6.5] provided strong support for this conjecture by computing 500 terms of $I(011,201)$ and $I(010,100,120,210)$. Finally,

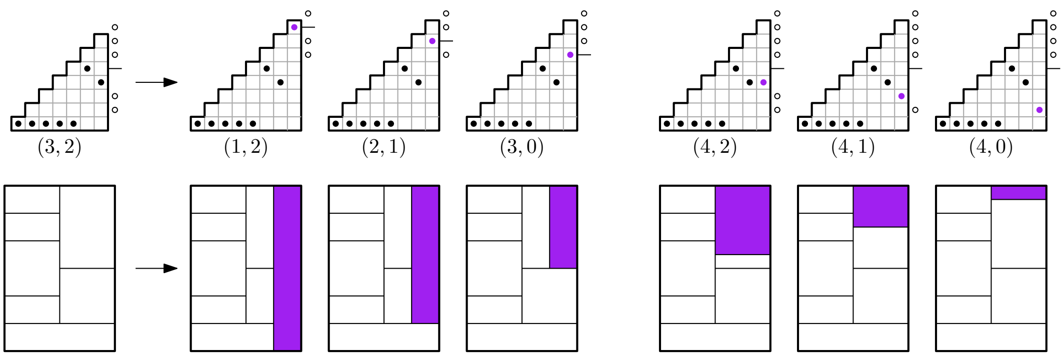
[1] Research of Andrei Asinowski was funded by the Austrian Science Fund (FWF) [10.55776/P32731].

[2] Research of Michaela Polley was funded by a Fulbright-Austrian Marshall Plan Foundation Award For Research In Science And Technology.

in [3, Table 1, Case 166], Callan and Mansour gave nine avoidance classes of inversion sequences determined by four patterns of length 3: four of them were confirmed to be enumerated by A279555 at the time of writing, while for five others (two of them were in fact alternative descriptions of $I(011, 201)$ and $(011, 210)$) it was still conjectured. In this contribution, we provide proofs for these conjectures.

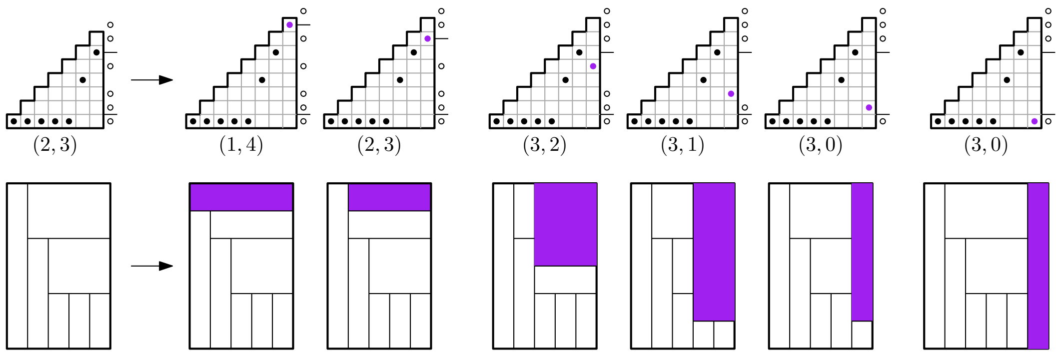
Bijections between $\top$-avoiding rectangulations and two classes of inversion sequences: $I(010, 101, 120, 201)$ and $I(011, 201)$. In our proofs, we use generating trees for $I(010, 100, 120, 210)$ and for $I(011, 201)$, given by Pantone [6, Section 6.5]. We show that both these trees generate the $\top$-avoiding rectangulations — hence, these classes of inversion sequences are equinumerous. In accordance with the remark above, for the former class we use the equivalent class $I(010, 101, 120, 201)$.

The generating tree that applies to $I(010, 101, 120, 201)$ and to $\top$-avoiding rectangulations, has root $(1, 0)$ and succession rules $(k, \ell) \rightarrow (1, k-1), (2, k-2), \ldots, (k, 0); (k+1, 0), (k+1, 1), \ldots, (k+1, \ell)$. For $e \in I(010, 101, 120, 201)$, k is the *bounce* — the difference between the size of e and its maximum value M; and ℓ is the number of values smaller than M that can be inserted at the end of e to produce a new sequence in the class. For a $\top$-avoiding rectangulation, k is the number of rectangles that touch the right side of R; and ℓ is the number of horizontal segments whose right endpoint touch the top-right rectangle. Inserting a new point at the end of e corresponds to inserting a new top-right rectangle into the corresponding rectangulation:



For the second construction, we use $\perp$-avoiding (instead of $\top$-avoiding) rectangulations. The generating tree that applies to $I(011, 201)$ and to $\top$-avoiding rectangulations, has root $(1, 0)$ and succession rules $(k, \ell) \longrightarrow (1, k+\ell-1), (2, k+\ell-2), \ldots, (k, \ell); (k+1, 0), (k+1, 1), \ldots, (k+1, \ell-1); (k+1, 0)$. For $e \in I(011, 201)$, k and ℓ have the same meaning as above with an additional condition that ℓ refers to *non-zero* admissible values. For a $\perp$-avoiding rectangulation, k is the number of rectangles that touch the top side of R; and ℓ is the number of $\top$ joints whose horizontal segment touches the right side of R. As in the first construction, inserting a new point at the end of e corresponds to inserting a new top-right rectangle into the corresponding rectangle (see the image on the next page).

The constructions sketched above yield bijections between both classes of inversion sequences and $\top$-avoiding rectangulations — hence, $I(010, 101, 120, 201)$ and $I(011, 201)$ are in bijection. Moreover, the numbers of rectangles that touch top, bottom, right, and left sides of R correspond to certain statistics in inversion sequences. Therefore, we obtain not only a bijection between two classes of inversion sequences, but also the following correspondence between quadruples of statistics:



Theorem. For every $n \geq 1$, we have $|I_n(010, 101, 120, 201)| = |I_n(011, 201)|$. Moreover, the quadruple of statistics (a, b, c, d) for $I_n(010, 101, 120, 201)$, where a is the number of elements that satisfy $e_j = 0$, b is the number of left-to-right-maxima, c is the bounce, and d is the number of elements that satisfy $e_j = j - 1$, matches the quadruple of statistics (x, y, z, t) for $I_n(011, 201)$, where x is the number elements that satisfy $e_j = j - 1$, y is the number of elements that satisfy $e_j = 0$, z is the number of right-left-minima, and t is the bounce.

We also construct direct bijections which align with the generating trees given above.

References

[1] Andrei Asinowski, Jean Cardinal, Stefan Felsner, and Éric Fusy. Combinatorics of rectangulations: Old and new bijections. Combinatorial Theory, 5:1 (2025).

[2] Andrei Asinowski and Michaela Polley. ⊤-avoiding rectangulations, inversion sequences, and Dyck paths. *Permutation Patterns 2024*.

[3] David Callan and Toufik Mansour. Inversion sequences avoiding quadruple length-3 patterns. Integers, 23 (2023), Article A78.

[4] Megan Martinez and Carla Savage. Patterns in inversion sequences II: Inversion sequences avoiding triples of relations. Journal of Integer Sequences, 21 (2018), Article 18.2.2.

[5] Arturo Merino and Torsten Mütze. Combinatorial generation via permutation languages. III. Rectangulations. Discrete and Computational Geometry, 70 (2023), 51–122.

[6] Jay Pantone. The enumeration of inversion sequences avoiding the patterns 201 and 210. Enumerative Combinatorics and Applications, 4:4 (2024), Article S2R25.

[7] Chunyan Yan and Zhicong Lin. Inversion sequences avoiding pairs of patterns. Discrete Mathematics & Theoretical Computer Science, 22:1 (2020), Article 23.