Geometric view of interval poset permutations

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This talk is based on joint work with Estrella Eisenberg, Shulamit Reches , and Moriah Sigron

Introduction

The notion of an interval poset for a permutation, initially introduced by Tenner in [5], effectively represents all intervals within a permutation and their inclusion relationships comprehensively.

Definition 1.1. The *interval poset of a permutation* $\pi \in \mathcal{S}_n$ is the poset $P(\pi)$ whose elements are the non-empty intervals of π; the order is defined by set inclusion and the minimal elements are the intervals of size 1.

Bouvel, Cioni and Izart [3], provided a formula for the number of interval posets with n minimal elements, sequence A348479 in [4]. Tenner enumerated binary interval posets. These are exactly the interval posets of the separable permutations. These posets are counted by the small Schröeder numbers. She also enumerated binary tree interval posets by $2C_{n-1}$ where C_n is the $n-$th Catalan number. Then, Bouvel, Cioni and Izart in [3] enumerated the tree interval posets (sequence A054515 in [4]). Interestingly enough, this sequence also enumerates the number of ways to place non-crossing diagonals in a convex $(n+1)$-gon such that no quadrilaterals are created.

In this study (see the full paper in [2]), we propose a different view on interval posets by introducing a simple bijection between the set of tree interval posets and the set of dissections of convex $(n+1)$-gons, satisfying the conditions listed above. We also use this bijection to enumerate interval posets of various subsets of permutations by using a wider set of polygons.

Interval posets and convex polygons

We identify a convex polygon with its set of vertices and denote a diagonal or an outer edge of the polygon from vertex i to vertex j by (i, j).

Let P be a convex $n-$ polygon. A *dissection* of P is a decomposition of it into finitely many sub-polygons.

Definition 2.1. A dissection of an $n-$ gon will be called *diagonally framed* if for each two crossing diagonals, all their vertices are connected to each other. Explicitly, if (a, c) and (b, d) are two crossing diagonals, then the diagonals or outer edges $(a, b), (b, c), (c, d), (d, a)$ must also exist (see Fig. 1 for an example).

We define a bijection between the set of interval posets with n minimal elements and the set of diagonally framed dissections of convex $(n+1)-$ gon without quadrilaterals as follows:

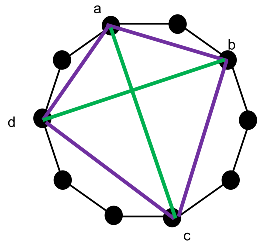

Figure 1: A 10 - gon with crossing diagonals (a, c) and (b, d) and their "frame".

Let P be the interval poset of some $\pi \in \mathcal{S}_n$. We set $\Phi(P)$ to be the convex $(n+1)-$gon whose set of diagonals is

$$\{(a, b+1) | [a, b] \text{is a non-minimal element of } P\},$$

i.e. to each interval of the form $[a, b]$ corresponds a diagonal $(a, b+1)$ in $\Phi(P)$; note that singleton intervals correspond to outer edges in the polygon (see Figure 2 for an example).

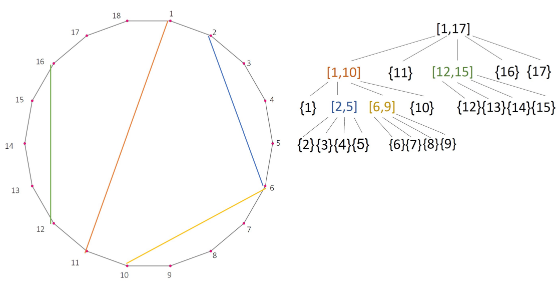

Figure 2: Right: the interval poset P. Left: the polygon $\Phi(P)$

Using the above bijection we proved the following results.

Theorem 2.2. The number of interval posets with n minimal elements is equal to the number of diagonally framed dissections of the convex $(n+1)$-gon such that no quadrilaterals are present

Theorem 2.3. The number of tree interval posets with n minimal elements is equal to the number of non-crossing dissections of the convex $(n+1)-$ gon such that no quadrilaterals are present.

In [1], the current authors introduced the notion of block-wise simple permutations. We cite here the definition:

Definition 2.4. A permutation $\pi \in \mathcal{S}_n$ is called *block-wise simple* if it has no interval of the form $p_1 \oplus p_2$ or $p_1 \ominus p_2$, where $\oplus$ and $\ominus$ stand for direct and skew sums of permutations respectively.

Theorem 2.5. The number of interval posets that represent a block-wise simple permutation of order n is equal to the number of ways to place non-crossing diagonals in a convex $(n+1)$-gon such that no triangles or quadrilaterals are present

The number of separable permutations of order n is known to be the (big) Schröeder number (sequence A006318 in the OEIS [4]) . In [5], Theorem 6.3, the author proved that each interval poset of a separable permutation corresponds to exactly two separable permutations, from which she concluded that the number of such interval posets is half this number. This last number is known as counting also the dissections of the $n+1$ - gon such that crossing diagonals are prohibited. (see [6]). Here we present a bijection that proves this connection in a combinatorial way.

Theorem 2.6. There is a bijection between the interval posets corresponding to separable permutations of order n and the set of dissections of the $n+1$ - gon such that crossing diagonals are prohibited.

References

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