Some integer values in the spectra of the Cayley graphs of colored permutations generated by prefix reversals

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This talk is based on joint work with Charles Buehrle

If G is a graph with adjacency matrix $A(G)$, then we refer to the set of eigenvalues of $A(G)$ along with their respective multiplicities as the *spectrum of* G. Cayley graphs are fundamental structures in algebra, and understanding their spectral properties sheds light on their structure. Some of these graphs have extremely nice spectra. For example, consider the so-called *Transposition graphs*, where two permutations π_1 and π_2 are connected by an edge if and only if there is a transposition t such that $t\pi_1 = \pi_2$. The transposition graph is *integral*, meaning that all of its eigenvalues are integers.

Another interesting Cayley graph is the so-called *Pancake Graph* P_n that arises from connecting permutations π_1 and π_2 if and only if there is a prefix reversal r_i such that $r_i\pi_1 = \pi_2$, with $1 < i \leq n$. Recall that a prefix-reversal is a permutation that in one-line-notation can be written as $r_i = i\ (i-1)\ \cdots\ 1\ (i+1)\ \cdots\ n$. In other words, r_i reverses the prefix $1\ 2\ 3\ \cdots\ i$ and leaves any element $j > i$ fixed. For example, in S_5, $r_3 = 32145$ and $r_5 = 54321$. The spectrum of P_n has been studied previously. For instance, Dalfó and Fiol [3] showed that the spectrum of P_n contains all elements from the set $[n-1] \cup \{-1, 0\}$.

Let us use $\mathbb{Z}_m$ to refer to the group of integers modulo m. We refer to the group $\mathbb{Z}_n \wr S_n$, where $\wr$ denotes the standard wreath product, as the group of *colored permutations* (referred to as index permutations by Steingrímsson in [4]). One can write the elements of $\mathbb{Z}_m \wr S_n$ as strings of the form $a_1^{e_1} a_2^{e_2} \cdots a_n^{e_n}$ with $a_i \in [n]$ and $e_i \in \mathbb{Z}_m$ for $1 \leq i \leq n$.

Consider $\pi_1^{e_1}\pi_2^{e_2}\cdots\pi_n^{e_n} \in \mathbb{Z}_m \wr S_n$ and $1 \leq i \leq n$. Then define

$$r_i(\pi_1^{e_1}\pi_2^{e_2}\cdots\pi_n^{e_n}) = \pi_i^{a_i}\pi_{i-1}^{a_{i-1}}\cdots\pi_1^{a_1}\pi_{i+1}^{e_{i+1}}\cdots\pi_n^{e_n}$$

where $a_j = e_j + 1 \mod m$. Notice that each r_i has an inverse given by

$$r_i^{-1}(\pi_1^{e_1}\pi_2^{e_2}\cdots\pi_n^{e_n}) = \pi_i^{b_i}\pi_{i-1}^{b_{i-1}}\cdots\pi_1^{b_1}\pi_{i+1}^{e_{i+1}}\cdots\pi_n^{e_n}$$

where $b_i = e_i - 1 \mod m$.

We furthermore refer to the set $R_n := \{r_i, r_i^{-1} : 1 \leq i \leq n\}$ as the set of *generalized prefix reversals*, or *prefix reversals* for simplicity.

A natural variation of pancake graphs is to use the colored permutations $\mathbb{Z}_m \wr S_n$ as set of vertices and the set $\{(s\pi_1, \pi_2) \mid s \in R_n \text{ and } \pi_1, \pi_2 \in \mathbb{Z}_m \wr S_n\}$. This graph is called the *prefix-reversal graphs*, which we denote by $P(m, n)$.

The case $m = 2$ is known as the *Burnt Pancake Graph*, denoted by BP_n. In [1], Blanco and Buehrle proved that the spectrum of BP_n contains all integers in the interval $[0, n] \setminus \{\lfloor n/2 \rfloor\}$.

In this work, the authors extend their result to all prefix reversal graphs with $m > 2$. More precisely, they establish the following theorem.

Theorem 1. *The spectrum of $P(m,n)$ contains all even integers in the set $[0,2n] \setminus \{2\lfloor n/2 \rfloor\}$.*

Furthermore, one can also consider the *directed prefix-reversal graph $DP(m,n)$*; that is, the graph with vertex set $\mathbb{Z}_m \wr S_n$ and edge set $\{(r_i\pi, \pi) \mid 1 \leq i \leq n, \pi \in \mathbb{Z}_m \wr S_n\}$. Notice that $DP(m,n)$ is obtained by not considering the inverses r_i^{-1} with $1 \leq i \leq n$. We prove that the spectrum of DP_n contains all integers from $[0,n] \setminus \{2\lfloor n/2 \rfloor\}$.

In other words, the spectra of all these prefix-reversal graphs contain a range of consecutive integers or even integers, respectively. This is not a common spectral property of a family of graphs.

In the case where $m = 4k$ for some positive integer k, we can prove that the entire range of even integers in $[0,2n]$ is included in the spectrum of $P(m,n)$ and that all integers in the interval $[0,n]$ are included in the spectrum of $DP(m,n)$.

One of the notions of the *spectral gap* of a graph G refers to the difference between the largest and second-largest eigenvalue of $A(G)$. Our results provide a bound for the spectral gap of these prefix reversal graphs, though the true spectral gap remains unknown still. The matter of finding the spectral gap for some of these prefix-reversal graphs was also addressed by Chung and Tobin in [2]. Although they did not provide a bound, they observed that the spectral gap seems to be getting smaller as $n \to \infty$. We are able to observe the same phenomenon in the spectra of $P(m,n)$ and $DP(m,n)$.

References

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