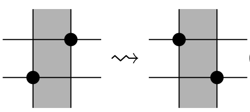
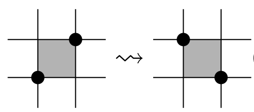
Between weak and Bruhat: the middle order on permutations

Mathilde Bouvel Université de Lorraine, CNRS, Inria, LORIA

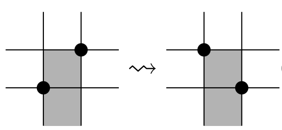
This talk is based on joint work with Luca Ferrari and Bridget E. Tenner

In this contribution, we are interested in partial orders that can be defined on the sets of permutations of any fixed size. Among the most classical such orders are the Bruhat order and the weak order, which motivate the context of our study. (We refer to [1] for background on the Bruhat and weak orders, and to [2] for basics on posets in general.)

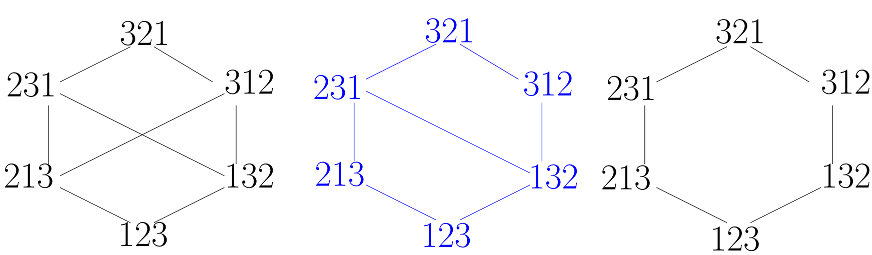
The Bruhat and weak orders are defined on the set S_n of all permutations of size n, and can be conveniently described in terms of the associated covering relations. More specifically, given $\sigma, \tau \in S_n$, the element σ is covered by τ in the Bruhat order when τ is obtained from σ by turning a noninversion into an inversion, provided that no further inversions are created. On the other hand, σ is covered by τ in the weak order when τ is obtained from σ by turning an ascent into a descent. These covering relations can therefore be summarized by the following transformations on occurrences of mesh patterns:

 (weak order), and  (Bruhat order).

We introduce a class of "intermediate" partial orders, which we call *middle orders*, that interpolate (with respect to refinement) between the Bruhat orders and the weak orders. Their covering relations correspond to the following transformations on occurrences of mesh patterns:

 (middle order).

By definition, the middle order refines the weak order on permutations and admits the Bruhat order as a refinement, justifying the terminology. This is illustrated in size 3 by the Hasse diagrams below.



Bruhat order middle order weak order

We demonstrate that there is a second combinatorial interpretation of this middle order, in terms of inversion sequences associated with permutations. In this context (and while symmetries are considered in other works), we define the inversion sequence of a permutation σ of size n as $I(\sigma) = (x_1, \ldots, x_n)$, where

$$x_i = \#\{j < i \mid \sigma^{-1}(j) > \sigma^{-1}(i)\}.$$

In other words, the ith coordinate of $I(\sigma)$ counts the times that the value i is an inversion top.

We show that σ is less than or equal to τ in the middle order if and only if $I(\sigma)$ is less than or equal to $I(\tau)$, coordinate-wise. As a consequence, as a poset, the middle order is isomorphic to the product of chains $[0,0] \times [0,1] \times [0,2] \times \cdots \times [0, n-1]$. Building from there, we can prove nice structural properties of the middle orders: they are distributive lattices, and they are graded.

We proceed by studying the intervals, and the Boolean intervals, of the middle order. We show that the number of intervals (resp. Boolean intervals) in the middle order for size n is $\frac{n!(n+1)!}{2^n}$ (resp. $(2n-1)!!$). Furthermore, denoting $f(n,k)$ (resp. $b(n,k)$) the number of intervals (resp. Boolean intervals) of rank k in the middle order for size n, we show that $f(1,0) = 1$ and for $n \geq 2$ and $k \in [0, \binom{n}{2}]$,

$$f(n,k) = \sum_{h=0}^{n-1} (n-h) \cdot f(n-1, k-h),$$

(with the convention that $f(n,j) = 0$ when $j < 0$), while we have a non-recursive formula in the Boolean case. Namely, denoting $c(n,j)$ the signless Stirling numbers of the first kind (counting permutations of size n having j cycles, or equivalently permutations of size n having j RtoL-minima), it holds that $b(n,k) = \sum_{i=0}^{n} \binom{i}{k} c(n, n-i)$ (here, k can only range from 0 to $n-1$).

With the good understanding of the Boolean intervals, and the property of being a distributive lattice, we have easy access to the Möbius function on the middle order (whereas the Möbius function is notoriously difficult to compute in general).

We are also able to give a combinatorial description of the Euler characteristic on the middle order, which counts the number of RtoL-non-minima of a permutation.

This talk and short abstract are based on our article [3].

References

[1] A. Björner and F. Brenti, *Combinatorics of Coxeter Groups*, Graduate texts in Mathematics, Springer, 2005.

[2] R. P. Stanley, *Enumerative Combinatorics, Volume 1*, Cambridge Studies in Advanced Mathematics 49, Cambridge University Press, Cambridge, 2012.

[3] M. Bouvel, L. Ferrari, B. E. Tenner, *Between weak and Bruhat: the middle order on permutations*, Graphs and Combinatorics, vol. 41, article number 34, 2025.