Finitely based classes below 2.618 are rational

Robert Brignall The Open University

This talk is based on joint work with Michal Opler.

The main result we will present in this talk is the following:

Theorem 1. *Every finitely based permutation class whose upper growth rate is less than $1 + \phi \approx 2.618$ is enumerated by a rational generating function.*

It was, in fact, conjectured at Permutation Patterns 2023 that the above theorem should hold up to growth rate 4. This conjecture is still open, and several elements of the methodology in this paper do not extend beyond $1 + \phi = (3 + \sqrt{5})/2$.

There are several steps to proving Theorem 1, the first of which is to prove a conjecture due to Vatter [3] concerning the concentration of cell classes below growth rate $1 + \phi$. A permutation class $\mathcal{C}$ is said to be *concentrated* if there exist integers q and r such that for any $\pi \in \mathcal{C}$, and for any horizontal or vertical line that slices the plot of π, there exists a collection of at most q axis-parallel rectangles in this plot such that: (1) No rectangle is sliced by the line. (2) The rectangles are pairwise *independent*. That is, for any two rectangles their projections onto both the horizontal and vertical axes are disjoint. (3) All but at most r points of π are contained in the rectangles.

For any positive real number γ, the *cell class* $\mathcal{G}_\gamma$ is defined by

$$\mathcal{G}_\gamma = \left\{\pi \;:\; \mathrm{gr}\left(\mathcal{S}_\pi^\oplus\right) < \gamma \text{ or } \mathrm{gr}\left(\mathcal{S}_\pi^\ominus\right) < \gamma\right\}.$$

where $\mathcal{S}_\pi^\oplus$ and $\mathcal{S}_\pi^\ominus$ denote the smallest sum-closed and skew-sum-closed permutation classes containing π, respectively.

Theorem 2. *The cell class $\mathcal{G}_\gamma$ is concentrated if and only if $\gamma < 1 + \phi$.*

The rest of the proof of Theorem 1 extends the grid class methodology of Vatter [2, 3], and then applies an extension of the insertion encoding (see Albert, Linton and Ruškuc [1]) to the resulting structure.

References

[1] Michael H. Albert, S. Linton, and N. Ruškuc. The insertion encoding of permutations. *Electron. J. Combin.*, 12(1): Research paper 47, 31 pp., 2005.

[2] Vincent Vatter. Small permutation classes. *Proc. Lond. Math. Soc. (3)*, 103: 879–921, 2011.

[3] Vincent Vatter. Growth rates of permutation classes: from countable to uncountable. *Proc. Lond. Math. Soc. (3)*, 119(4): 960–997, 2019.