Wilf equivalence of partial permutations

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Hamaker and Rhoades [2] define a *partial permutation* (A, B) *of size* k as an ordered pair of lists $A = (a_1, \ldots, a_k)$ and $B = (b_1, \ldots, b_k)$ of positive integers, representing a map $\pi : A \to B$ such that $\pi(a_i) = b_i$ for all $i = 1, \ldots, k$. Pairs (A, B) that represent the same map π are considered to be the same partial permutation.

For any $k \geq m$, a *k-extension* of a partial permutation (A, B) of size j is a permutation $\sigma \in S_k$ such that $\sigma(a_i) = b_i$ for each $i = 1, \ldots, m$. In other words, $\sigma|_A = \pi$. A *k-completion* $(A, B)_k$ of a partial permutation (A, B) is the set of all k-extensions of (A, B).

We call (A, B) and (C, D) *k-Wilf-equivalent as partial permutations* if $(A, B)_k \sim (C, D)_k$ (where "$\sim$" denotes Wilf-equivalence). Furthermore, we call (A, B) and (C, D) *Wilf-equivalent as partial permutations* if $(A, B)_k \sim (C, D)_k$ for all $k \geq \max(A \cup B \cup C \cup D)$. We denote this by $(A, B) \sim (C, D)$.

We prove some and conjecture other Wilf-equivalences of partial permutations, and extend some of those to *shape-Wilf-equivalences* of partial permutations. Many of those can be restated as (shape-)Wilf-equivalences of partially ordered permutations (POPs), proved or conjectured in Burstein et al. [1]. One of those conjectures was recently proved by Wang and Yan [3].

We also establish some enumerative results. For example:

Theorem 1. *We have* $((t, t+1), (1, 2))_k \sim (12, 12)_k$ *for all* $1 \leq t \leq k-1$, *and therefore* $((t, t+1), (1, 2))$, $t \geq 1$, *are Wilf-equivalent as partial permutations. Moreover, for* $k \geq 3$,

$$|\operatorname{Av}_n\left(((t, t+1), (1, 2))_k\right)| = \begin{cases} n!, & \text{if } n < k-3, \\ (k-3)!\, r_{k-3}(n), & \text{if } n \geq k-3, \end{cases}$$

where $r_{k-3}(n)$ *is the* n*-th* $(k-3)$*-Schröder number, i.e. the number of Schröder paths with level steps of* $k-3$ *colors.*

References

[1] A. Burstein, T. Han, S. Kitaev, P. Zhang. On (shape-)Wilf-equivalence of certain sets of (partially ordered) patterns, *Electronic Journal of Combinatorics* **32(1)** (2025), #P1.7. https://doi.org/10.37236/13037

[2] Z. Hamaker, B. Rhoades. Partial permutations and character evaluations, preprint. https://arxiv.org/pdf/2503.17552

[3] L. Wang, S.H.F. Yan. Proof of a conjecture on the shape-Wilf-equivalence for partially ordered patterns. https://arxiv.org/pdf/2503.22098