On inversions, major index, and the area statistic on ℓ-interval parking functions

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This talk is based on joint work with Jennifer Elder, Kimberly P. Hadaway, Pamela E. Harris, Jeremy L. Martin, Amanda Priestley, and Gabe Udell

Consider the following parking scenario. There are n cars attempting to park in n spots on a one way street, and their preferred parking spot of each car is recorded as a tuple $\alpha = (a_1, a_2, \ldots, a_n) \in [n]^n$. As each car parks, it drives to its preferred spot and parks there if it is available. If that spot is not available, the car continues driving and parks in the next available spot. If all the cars are able to park in their preferred spot, or at most ℓ away from that spot, we call α an *ℓ-interval parking function*. Let $\mathrm{IPF}_n(\ell)$ denote the class of ℓ-interval parking functions of length n. These were introduced by Aguilar-Fraga et al. [1] generalizing work of Colaric et al. in [6].

Enumerative results on ℓ-interval parking functions with $\ell \geq 1$ also give connections between this set of parking functions and Dyck paths with restricted height and to preferential arrangements [1]. However, one aspect that is not well understood is the discrete statistics of these combinatorial objects. In this talk, we study inversion, major index and area statistics on the set of ℓ-interval parking functions.

We start with a natural question: Are inversion and major index equidistributed on the set of ℓ-interval parking functions? Clearly, the answer is "yes" for $\ell = 0, n-1$ since these are permutations and parking functions, respectively. But what about the ℓ in between? We provide a complete answer: first, recall the classical *Foata transform* on words turns a word w into a word w' such that $\mathrm{inv}(w) = \mathrm{maj}(w')$.

Theorem 1. *Let* $1 \leq \ell \leq n-2$.

1. *If* $\ell \in \{1, 2, n-2\}$*, then the Foata transform restricts to a bijection* $\mathrm{IPF}_n(\ell) \to \mathrm{IPF}_n(\ell)$.
2. *If* $\ell \notin \{1, 2, n-2\}$*, then inversion and major index are not equidistributed on* $\mathrm{IPF}_n(\ell)$.

The $\ell = 1$ in fact follows from the existence of $\mathfrak{S}_n$-action on $\mathrm{IPF}_n(1)$ [5]. Interestingly, there does not appear to be any $\mathfrak{S}_n$-action for $\ell = 2, n-2$.

Next, we use area to give enumerative formulas for ℓ-interval parking functions. For a permutation $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in \mathfrak{S}_n$ and index $i \in [n]$, define

$$L_\ell(i;\sigma) = \min(\ell + 1, i - t + 1) \tag{1}$$

where $\sigma_t, \sigma_{t+1}, \ldots, \sigma_i$ is the longest contiguous subsequence of σ such that $\sigma_k \leq \sigma_i$ for all $k \in [t, i]$. The case $\ell = n-1$ (which includes all parking functions, without the ℓ-interval restriction) was studied in [7]. Recall that the *area* of a parking function $\alpha \in \mathrm{PF}_n$ is the quantity $\mathrm{area}(\alpha) = \sum_{i=1}^{n} i - \alpha_i$.

Proposition 2 ([7] $\ell = n-1$)**.** *Let $0 \le \ell \le n-1$. For all $\sigma \in \mathfrak{S}_n$, we have*

$$\sum_{\alpha \in \mathrm{IPF}_n(\ell)} q^{\mathrm{area}(\alpha)} = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^{n} [L_\ell(i;\sigma)]_q.$$

We call a 1-interval parking function a *unit* interval parking function and denote $\mathrm{UPF}_n = \mathrm{IPF}_n(1)$. This set is enumerated by the Fubini numbers [4, 10] and unit interval parking functions has been shown to be connected to many other famous combinatorial objects such as the combinatorial game of the Tower of Hanoi [2], the facets of the permutahedron [5], and Boolean intervals in the weak Bruhat order of the symmetric group [9] to name a few. We can refine our results for unit interval parking functions by considering are and inversions simulateously.

Corollary 3. *For all $n \ge 1$, $\sum_{\alpha \in \mathrm{UPF}_n} q^{\mathrm{area}(\alpha)} t^{\mathrm{inv}(\alpha)} = \sum_{\sigma \in \mathfrak{S}_n} (1+q)^{\mathrm{asc}(\sigma)} t^{\mathrm{inv}(\sigma)}$.*

Let X be a finite set, let $X(q)$ be a polynomial in q such that $X(1) = |X|$, let $C = \langle g \rangle$ be a cyclic group of order n acting on X, and let ω be a primitive complex nth root of unity. The triple $(X, X(q), C)$ is said to exhibit the *cyclic sieving phenomenon* (CSP) [11] if $X(\omega^j) = |\{x \in X : c^j(x) = x\}|$ for every $c \in C$.

As shown in [5, Lemma 3.12, Proposition 3.13, Corollary 3.14], there is an $\mathfrak{S}_n$-action on UPF_n that fixes area. The cyclic subgroup C_n of $\mathfrak{S}_n$ generated by the cycle $g = (1, 2, \dots, n) \in \mathfrak{S}_n$ acts on UPF_n by restricting the $\mathfrak{S}_n$-action on UPF_n to C_n. For $k \in \mathbb{N}$, let $\mathrm{UPF}_{n,k}$ be the set of unit interval parking functions with area k and set

$$f_{n,k}(t) = \sum_{\alpha \in \mathrm{UPF}_{n,k}} t^{\mathrm{inv}(\alpha)}.$$

Theorem 4. *The triple $(\mathrm{UPF}_{n,k}, f_{n,k}(t), C_n)$ exhibits CSP for each n and k.*

Finally, the *Lehmer code* of a permutation σ is the sequence $L(\sigma) = (a_1, \dots, a_n)$ so that $a_i = \#\{j < i \mid \sigma^{-1}(i) > \sigma^{-1}(j)\}$. The map $\sigma \mapsto L(\sigma)$ defines a bijection between $\mathfrak{S}_n$ and the set $E_n := \prod_{i=0}^{n-1}[0, i]$. Unit interval parking functions also have a notion of Lehmer code: a *cipher* is a pair (w, I) of $w \in E_n$ and a selection of positions $I = \{i_1, \dots, i_k\}$ such that $\mathrm{Asc}(w) \subseteq I$, viewing w as a word. Let $\mathcal{G}_n$ denote the set of ciphers of length n. The following theorem is a consequence of work by Avalos and Bly [3].

Theorem 5. *There is a bijection $\psi : \mathrm{UPF}_n \to \mathcal{G}_n$ such that for each $\alpha \in \mathrm{UPF}_n$, if $\psi(\alpha) = (w, I)$, then $\mathrm{inv}(\alpha) = \sum_{i=1}^{n} w_i$, $|I| = \mathrm{area}(\alpha)$ and if σ_i is the car parking in spot i, then $L(\sigma_1 \cdots \sigma_n) = w$.*

Ciphers are a great way of understanding unit interval parking functions. For instance, they can very quickly allow you to derive a new proof of the following.

Corollary 6 ([9, Theorem 1.2])**.** *Unit interval park functions with j inversions and $n-k$ blocks all of size 1 or 2 are in bijection with rank k boolean intervals $[u, v]$ of $\mathfrak{S}_n$ under the right weak order with $\mathrm{inv}(u) = j$.*

To conclude, we discuss a number of different statistics that we would like to know more about for ℓ-interval parking functions.

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