Descents of permutations with only even or only odd cycles

Sergi Elizalde Dartmouth College

Let $\mathcal{S}_n^o$ denote the set of permutations in $\mathcal{S}_n$ all of whose cycles have odd length. Let $\mathcal{S}_n^e$ be the set of permutations all of whose cycles have even length, except possibly for one cycle of length one (i.e., a fixed point). It is known that $|\mathcal{S}_n^o| = |\mathcal{S}_n^e|$ for all n, as one can show using exponential generating functions. When n is even, a bijective proof appears in Bóna's book [2, Lem. 6.20], and it is not hard to extend it to odd n.

In a recent preprint [1], Adin, Hegedűs and Roichman proved that that the identity $|\mathcal{S}_n^o| = |\mathcal{S}_n^e|$ has a surprising refinement. For $\pi \in \mathcal{S}_n$, denote its ascent and descent sets by $\mathrm{Asc}(\pi) = \{i \in [n-1] : \pi_i < \pi_{i+1}\}$ and $\mathrm{Des}(\pi) = \{i \in [n-1] : \pi_i > \pi_{i+1}\}$.

Theorem 1 ([1])**.** *For any n and any subset $J \subseteq [n-1]$,*

$$|\{\pi \in \mathcal{S}_n^o : \mathrm{Asc}(\pi) = J\}| = |\{\pi \in \mathcal{S}_n^e : \mathrm{Des}(\pi) = J\}|.$$

The proof in [1] relies on a new generating function identity on higher Lie characters, so a natural question is whether Theorem 1 has a bijective proof. Unfortunately, Bóna's bijection between $\mathcal{S}_n^o$ and $\mathcal{S}_n^e$ does not behave well with respect to descent sets.

Our main result is a bijective proof of Theorem 1. More specifically, we prove the following, from where Theorem 1 follows by the principle of inclusion-exclusion.

Theorem 2. *For any n and any subset $S \subseteq [n-1]$, there exists an explicit bijection*

$$f : \{\pi \in \mathcal{S}_n^o : \mathrm{Asc}(\pi) \subseteq S\} \to \{\pi \in \mathcal{S}_n^e : \mathrm{Des}(\pi) \subseteq S\}. \tag{1}$$

Our proof combines two known bijections between permutations and multisets of necklaces, together with a new bijection for Lyndon factorizations of words. We summarize the main ideas below. For more details and examples, we refer the reader to [3].

Let $S = \{s_1, s_2, \dots, s_{k-1}\}_< \subseteq [n-1]$. Denote its associated composition by $\alpha = \alpha(S) = (s_1, s_2 - s_1, \dots, s_{k-1} - s_{k-2}, n - s_{k-1})$, and define the monomial $\mathbf{x}^\alpha = \prod_{i=1}^k x_i^{\alpha_i}$. Fix a totally ordered alphabet $A = \{a_1, a_2, \dots, a_k\}$. Denote by $\mathcal{W} = A^*$ the set of finite words over A, and by $\mathcal{W}_n$ the set of those of length n. Two words $u, v \in \mathcal{W}$ are *conjugate* if they are cyclic rotations of each other. A *necklace* is a conjugacy class of words in $\mathcal{W}$. A word u is *primitive* if it is not the power of another word. A necklace is *primitive* if it is the conjugacy class of a primitive word.

Let $\mathcal{M}_n$ be the set consisting of all multisets of primitive necklaces of total length n. The *cycle structure* of $M \in \mathcal{M}_n$ is the partition of n whose parts are the lengths of the necklaces in the multiset, and its *weight* is the monomial $\mathrm{wt}(M) = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_k^{\alpha_k}$, where α_i is the number of times that a_i appears in M. Define the *weight* of a word $w \in \mathcal{W}$ similarly. Let $\mathcal{M}_n^o$ be the set of elements in $\mathcal{M}_n$ which consist of *distinct* necklaces of *odd* length, and let $\mathcal{M}_n^e$ be the set of those which consist of necklaces of *even* length, except possibly for one necklace of length one.

We start by applying a bijection due to Gessel and Reutenauer [5, Lem. 3.4] to the right-hand side of equation (1). Even though the original bijection is defined on words, we interpret it as a map from permutations to multisets of necklaces that preserves the cycle structure. When applied to $\mathcal{S}_n^e$, we get the following.

Proposition 3. *There exists a bijection*

$$\Phi_S : \{\pi \in \mathcal{S}_n^e : \mathrm{Des}(\pi) \subseteq S\} \to \{M \in \mathcal{M}_n^e : \mathrm{wt}(M) = \mathbf{x}^{\alpha(S)}\}.$$

The bijection Φ_S does not work well with permutations whose ascent set is contained in S. Instead, to deal with the left-hand side of equation (1), we use a different bijection that, in a slightly different form, has appeared in work of Gessel, Restivo and Reutenauer [4, Sec. 3], and is also a special case of a bijection due to Steinhardt [7]. In general, the necklaces in the image of this bijection may not be primitive or distinct, but they are when we restrict to $\mathcal{S}_n^o$.

Proposition 4. *There exists a bijection*

$$\Xi_S : \{\pi \in \mathcal{S}_n^o : \mathrm{Asc}(\pi) \subseteq S\} \to \{M \in \mathcal{M}_n^o : \mathrm{wt}(M) = \mathbf{x}^{\alpha(S)}\}.$$

Consider the lexicographic order on $\mathcal{W}$, which we denote by $<$. A primitive word in $\mathcal{W}$ is called a *Lyndon word* if it is strictly smaller than all the other words in its conjugacy class; equivalently, if it is strictly smaller than all of its proper suffixes [6, Prop. 5.1.2]. Lyndon words are in one-to-one correspondence with primitive necklaces, since each conjugacy class of primitive words has a unique lexicographically smallest element. Denote by $\mathcal{L}$ the set of Lyndon words in $\mathcal{W}$. The following is a well-known result of Lyndon.

Theorem 5 ([6, Thm. 5.1.5])**.** *Every $w \in \mathcal{W}$ has a unique* Lyndon factorization, *that is, an expression $w = \ell_1\ell_2\ldots\ell_m$ where $\ell_i \in \mathcal{L}$ for all i, and $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_m$.*

Identifying primitive necklaces with Lyndon words allows us to view multisets of primitive necklaces as Lyndon factorizations of words: each necklace in $M \in \mathcal{M}_n$ becomes a Lyndon factor of the associated word $w \in \mathcal{W}_n$.

Let $\mathcal{W}_n^o$ be the set of words in $\mathcal{W}_n$ all of whose Lyndon factors have *odd* length and are *distinct*. Let $\mathcal{W}_n^e$ be the set of words in $\mathcal{W}_n$ all of whose Lyndon factors have *even* length, except possibly for one factor which has length one. The above identification gives straightforward bijections between $\mathcal{M}_n^o$ and $\mathcal{W}_n^o$, and between $\mathcal{M}_n^e$ and $\mathcal{W}_n^e$. Using Propositions 3 and 4, Theorem 2 is now equivalent to the following.

Theorem 6. *There exists a weight-preserving bijection $\Psi : \mathcal{W}_n^o \to \mathcal{W}_n^e$.*

To describe the map Ψ, we need the one more definition. The *standard factorization* of a Lyndon word $w \in \mathcal{L} \setminus A$ is the expression $w = rs$ where s is the longest proper suffix of w that belongs to $\mathcal{L}$, or equivalently, the lexicographically smallest proper suffix of w. It is known [6, Prop. 5.1.3] that, in this case, $r \in \mathcal{L}$ and $r < rs < s$.

Given $w \in \mathcal{W}_n^o$, we build $\Psi(w)$ by repeatedly applying certain updates to a pair of words (O, E). Initially, $(O, E) = (w, -)$, where $-$ denotes the empty word. Each step moves some subword from O to the beginning of E. At any time, all the Lyndon factors of O are odd and distinct, and all the Lyndon factors of E are even. At the end of the algorithm, we have $(O, E) = (-, \Psi(w))$.

Definition 7 (The bijection Ψ)**.** On input $w \in \mathcal{W}_n^o$, initially set $(O, E) = (w, -)$, and iterate the following step as long as $|O| \geq 2$:

- Let $O = o_1 o_2 \ldots o_m$ be the Lyndon factorization of O. Say that o_m is *splittable* if $|o_m| \geq 2$ and its standard factorization $o_m = rs$ satisfies $s < o_{m-1}$ (if $m = 1$, then o_m is splittable by convention). Update (O, E) to

$$(O', E') = \begin{cases} (o_1 o_2 \ldots o_{m-1} r,\ sE) & \text{if } o_m \text{ is splittable and } r \text{ has odd length,} \\ (o_1 o_2 \ldots o_{m-1} s,\ rE) & \text{if } o_m \text{ is splittable and } r \text{ has even length,} \\ (o_1 o_2 \ldots o_{m-2},\ o_m o_{m-1} E) & \text{if } o_m \text{ is not splittable.} \end{cases}$$

If we reach $|O| = 1$ (this case only occurs when n is odd), move this letter to the Lyndon factorization of E by inserting it as a new factor, in the unique location that keeps the factors weakly decreasing from left to right.

Once O is empty, let $\Psi(w) = E$.

With some work, we can show that Ψ is a weight-preserving bijection $\Psi : \mathcal{W}_n^o \to \mathcal{W}_n^e$, and therefore the composition $f = \Phi_S^{-1} \circ \Psi \circ \Xi_S$ proves Theorem 2.

References

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