Pattern avoidance in restricted-growth words of type B

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This talk is based on joint work with Eli Bagno, Toufik Mansour, and Amir Safadi

In the last decades, the area of pattern avoidance on set partitions of type A has received a lot of attention (see [11] and references therein). In particular, Sagan [15] offered two different approaches to pattern-avoidance in set partitions of type A, one of them, following Klazar [8, 9], was based on avoiding some structures of blocks of the set partitions (where the blocks are ordered increasingly with respect to their minimal elements), while the other used their restricted-growth (RG-)words representation. Sagan [15, Prop. 4.1] proved that pattern-avoidance in the set partitions sense implies pattern-avoidance in the RG-words representation sense, but not the other way around.

It seems that the first interest in pattern-avoidance in set partitions of type B arose in the seminal paper of Reiner [14], who introduced the notion of set partitions of type B in such a way that it corresponds to the plane arrangement of the Coxeter group of type B. Athanasiadis [1] calculated the number of noncrossing and nonnesting partitions of fixed block sizes for types B and C and for most of the partitions of type D. His work was completed by Athanasiadis and Reiner in [2].

Most of the results presented in this research were obtained using the method of *generating trees*, presented briefly below: Any set $\mathcal{C}$ of discrete objects with a notion of a size such that for each n, there are finitely many objects of this size is called a *combinatorial class*. West [16] introduced the notion of a *generating tree* for $\mathcal{C}$, which is a rooted, labeled tree whose vertices are the objects of $\mathcal{C}$ with the following properties:

1. Each object of $\mathcal{C}$ appears exactly once in the tree.
2. Objects of size n appear at level n in the tree (the root has level 0).
3. The children of every object are obtained by a set of succession rules in a form that determines the number of children and their labels.

This idea has been further utilized in closely related problems, see e.g. in [4, 5, 6, 7]. This technique has been used to count pattern-avoiding classes of some combinatorial objects, see for example [10, 12].

In this research, we make use of this method to count pattern-avoidance classes of RG-words of type B. We introduced the requested definitions:

Definition 1 ([3]). Let Σ^B be the alphabet $\{0, \pm 1, \pm 2, \ldots, \pm n\}$ and define the following order on Σ^B: $0 \prec -1 \prec 1 \prec -2 \prec 2 \prec \cdots \prec -n \prec n$.

A *restricted-growth (RG-)word of type B of the second kind of length n* is a word $\omega = \omega_1 \cdots \omega_n$ in the alphabet Σ^B which satisfies the following conditions:

(1) We have $\omega_1 = 0$ or $\omega_1 = 1$.

(2) For each $2 \leq t \leq n$, the following inequality holds:

$$\omega_t \preceq \max\{|\omega_1|, \dots, |\omega_{t-1}|\} + 1,$$

with respect to the order defined above.
In the case that

$$|\omega_t| = \max\{|\omega_1|, \dots, |\omega_{t-1}|\} + 1, \tag{1}$$

we demand: $\omega_t > 0$.

Denote by $R^B(n,k)$ the set of RG-words of type B of length n whose maximal element is k. Moreover, define $\mathcal{R}_n^B = \bigcup_k R^B(n,k)$ and $\mathcal{R}^B = \bigcup_{n \geq 0} \mathcal{R}_n^B$.

Let $\omega = \omega_1 \cdots \omega_n \in \mathcal{R}_n^B$ and let $\tau = \tau_1 \cdots \tau_k$ for some $1 \leq k \leq n$ be any word over the alphabet Σ^B such that $\{|\tau_1|, \dots, |\tau_k|\} = \{0, 1, \dots, d\}$ for some $d \leq n$. We say that ω *contains* τ, if there is a sequence of k indices, $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $\omega_{i_a} X \omega_{i_b}$ if and only if $\tau_a X \tau_b$, for all $1 \leq i < j \leq k$ and $X \in \{<, =, >\}$. In such a context, τ is usually called a *signed pattern*. For example, the RG-word of type B: $001(-1)(-1)21$ avoids 210 but contains 100 (due to the subsequence $1(-1)(-1)$). We denote the set of all τ-avoiding RG-words of signed set partitions in $\mathcal{R}_n^B$ by $\mathcal{R}_n^B(\tau)$. For an arbitrary finite collection of patterns P, we say that ω avoids P if ω avoids each $\tau \in P$; we denote the corresponding subset of $\mathcal{R}_n^B$ by $\mathcal{R}_n^B(P)$. We say that two sets of patterns P and P' are Wilf-equivalent if $\left|\mathcal{R}_n^B(P)\right| = \left|\mathcal{R}_n^B(P')\right|$, for all $n \geq 0$. Let P be any set of nonempty patterns. Define $\mathcal{R}^B(P) = \bigcup_{n=0}^{\infty} \mathcal{R}_n^B(P)$.

Inspired by Mansour and Safadi [12], we construct a generating tree $\mathcal{T}(P)$ for the class of pattern-avoiding RG-words of signed set partitions $\mathcal{R}^B(P)$. Starting with the root ϵ which occupies level 0, we recursively construct additional nodes of the tree $\mathcal{T}(P)$ such that the nth level of the tree consists of exactly the elements of $\mathcal{R}_n^B(P)$ arranged so that the parent of an RG-word $\omega_1 \cdots \omega_n \in \mathcal{R}_n^B(P)$ is the unique RG-word $\omega_1 \cdots \omega_{n-1} \in \mathcal{R}_{n-1}^B(P)$. The children of $\omega_1 \cdots \omega_{n-1} \in \mathcal{R}_{n-1}^B(P)$ are all the elements $\omega_1 \cdots \omega_{n-1}\omega_n$ from the set of the RG-words of type B of length n, satisfying the pattern-avoiding restrictions of the patterns in P. Figure 1 presents the first few levels of $\mathcal{T}(\{01\})$. Clearly, the cardinality of $\mathcal{R}_n^B(P)$ is the number of nodes in the nth level of $\mathcal{T}(P)$.

The main achievement of this research is a classification of pattern-avoidance for a single pattern of length 2 and 3.

For patterns of length 2, we have the following result:

Theorem 2. For all $n \geq 0$,

(1) $\sum_{n \geq 0} \left|\mathcal{R}_n^B(00)\right| \frac{x^n}{n!} = (1+x)e^{x + \frac{x^2}{2}}$,

(2) $\sum_{n \geq 0} \left|\mathcal{R}_n^B(01)\right| x^n = \frac{1-x+x^2}{(1-x)^3}$,

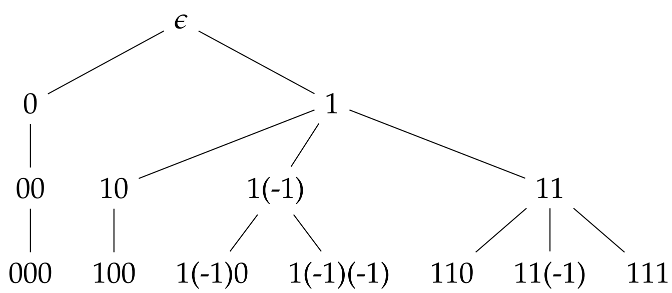

Figure 1: The generating tree $\mathcal{T}(\{01\})$

(3) $\sum_{n\geq 0} \left|\mathcal{R}_n^B(10)\right| x^n = \frac{1}{1-2x}$,

For patterns of length 3, Table 1 presents the first terms of the sequence $\{|\mathcal{R}_n^B(\tau)|\}_{n=1}^9$, for all such patterns τ ordered by τ; none of them appears in the OEIS [13]. We denote o✓ (e✓, resp.) where we have computed the ordinary (exponential, resp.) generating function. It is noteworthy that some of the proofs use combinatorial arguments.

τ	$\{\|\mathcal{R}_n^B(\tau)\|\}_{n=1}^9$	Enumeration
000	2, 6, 22, 98, 486, 2692, 16346, 107382, 756748	
001, 010	2, 6, 22, 87, 357, 1517, 6677, 30407, 143027	
011	2, 6, 22, 87, 361, 1554, 6907, 31609, 148664	e✓
012	2, 6, 22, 84, 315, 1138, 3941, 13093, 41857	o✓
021	2, 6, 22, 86, 339, 1322, 5069, 19084, 70583	o✓
100	2, 6, 22, 92, 432, 2224, 12392, 74064, 470944	e✓
101, 110	2, 6, 22, 91, 412, 2002, 10306, 55709, 314146	
102, 120	2, 6, 22, 85, 330, 1276, 4916, 18901, 72602	o✓
201, 210	2, 6, 23, 100, 467, 2285, 11559, 59960, 317201	o✓

Table 1: The sequence $\{|\mathcal{R}_n^B(\tau)|\}_{n=1}^9$, for all patterns τ of length three.

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