Positional statistics for separable permutations

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This talk is based on joint work with Oscar Lopez and Mike Weiner

Using generating trees, West [1] showed that the class $\mathcal{S}(2413, 3142)$ of separable permutations is counted by the large Schröder numbers. In this talk, we will discuss how to enumerate these permutations by the absolute position of the 1 (in one-line notation), and by the position of the 1 relative to the position of the maximal entry. We will show that, as suspected by West, enumeration by the position of the 1 leads to an alternative way to arrive at the Schröder numbers.

Let

$$S(x) = 1 + x + 2x^2 + 6x^3 + 22x^4 + 90x^5 + 394x^6 + \cdots$$

be the generating function for the sequence $a_0 = 1$, $a_n = |\mathcal{S}_n(2413, 3142)|$, and let $\mathcal{S}_n^{\ell\mapsto 1}$ be the subset of elements in $\mathcal{S}_n(2413, 3142)$ having the 1 at position ℓ. That is,

$$\mathcal{S}_n^{\ell\mapsto 1} = \{\sigma \in \mathcal{S}_n(2413, 3142) : \sigma(\ell) = 1\}.$$

Proposition 1. *The generating function* $g(x,u) = \sum_{n=1}^{\infty}\sum_{\ell=1}^{n} |\mathcal{S}_n^{\ell\mapsto 1}|\, u^\ell x^n$ *satisfies*

$$g(x,u) = \frac{xuS(x)S(xu)}{S(x) + S(xu) - S(x)S(xu)}.$$

Proof. Let $g_i(x,u)$ and $g_d(x,u)$ be the components of $g(x,u)$ counting the corresponding indecomposable and decomposable permutations of size greater than 1. Thus,

$$\begin{aligned} g_i(x,u) &= u^2x^2 + (u^2 + 2u^3)x^3 + \cdots, \\ g_d(x,u) &= ux^2 + (2u + u^2)x^3 + \cdots, \end{aligned}$$

and $g(x,u) = xu + g_i(x,u) + g_d(x,u)$. Note that, since every permutation is either indecomposable or has an indecomposable factor, we have

$$g(x,u) = (xu + g_i(x,u))S(x) \;\text{ and }\; g_d(x,u) = (xu + g_i(x,u))(S(x) - 1). \tag{1}$$

Since the reverse map is an involution on $\mathcal{S}_n(2413, 3142)$, we have

$$g(x,u) = ug(xu, \tfrac{1}{u}) = (xu + ug_i(xu, \tfrac{1}{u}))S(xu).$$

On the other had, since the reverse of an indecomposable permutation is decomposable, we also have $g_d(x,u) = ug_i(xu, \frac{1}{u})$, and therefore

$$\begin{aligned} g(x,u) &= (xu + g_d(x,u))S(xu) \\ &= xuS(x)S(xu) + g_i(x,u)(S(x) - 1)S(xu). \end{aligned}$$

Combining this with (1), we arrive at the equation

$$(xu + g_i(x,u))S(x) = xuS(x)S(xu) + g_i(x,u)(S(x) - 1)S(xu),$$

which gives

$$g_i(x,u) = \frac{xuS(x)(S(xu)-1)}{S(x)+S(xu)-S(x)S(xu)}.$$

Using again (1), we then get

$$g(x,u) = (xu + g_i(x,u))S(x) = \frac{xuS(x)S(xu)}{S(x)+S(xu)-S(x)S(xu)}.$$

□

Remark 2. Lettting $u=1$, Proposition 1 gives the equation

$$g(x,1) = S(x) - 1 = \frac{xS(x)S(x)}{S(x)+S(x)-S(x)S(x)} = \frac{xS(x)}{2-S(x)},$$

which leads to $S(x) = \frac{1}{2}\big(3 - x - \sqrt{1-6x+x^2}\big)$.

Enumeration by relative position

For $a,k \geq 1$, let $\mathcal{S}_{n,k}^{a\prec n}(2413,3142)$ be the set of $\sigma \in \mathcal{S}_n(2413,3142)$ such that:

- $\sigma^{-1}(n) - \sigma^{-1}(a) = k$,
- $\sigma^{-1}(b) - \sigma^{-1}(n) > 0$ for every $b \in \{1,\dots,a-1\}$.

For $a > 1$, any permutation $\sigma \in \mathcal{S}_{n,k}^{a\prec n}(2413,3142)$ must be of the form $\sigma = \pi \ominus \tau$, where $\pi \in \mathcal{S}_{m,k}^{1\prec m}(2413,3142)$ with $m = n-a+1$, and $\tau \in \mathcal{S}_{a-1}(2413,3142)$. For this reason, it is enough to just examine the case $a=1$.

Theorem 3. *The generating function* $f(x,t) = \sum_{n,k\geq 1} |\mathcal{S}_{n,k}^{1\prec n}(2413,3142)|t^k x^n$ *satisfies*

$$f(x,t) = \frac{x^2 t S(x) S(xt)^2}{\big(S(xt)+S(x)-S(xt)S(x)\big)^2}.$$

As a consequence, $\sum_{n,k,a\geq 1} |\mathcal{S}_{n,k}^{a\prec n}(2413,3142)|t^k s^a x^n = f(x,t)\cdot sS(xs)$.

References

[1] J. West, Generating trees and the Catalan and Schröder numbers, *Discrete Math.* **146** (1995), 247–262.