Proofs of Two Conjectures Mentioned in Moscow, USA

Anant Godbole East Tennessee State University

Let π_n be a uniformly chosen random permutation on $[n]$. The authors of Allen et al. [1] showed that the expected number of distinct *consecutive* patterns of all lengths $k \in \{1, 2, \ldots, n\}$ in π_n was $\frac{n^2}{2}(1 - o(1))$ as $n \to \infty$, exhibiting the fact that random permutations pack *consecutive* patterns near-perfectly. A conjecture was made in a 2018 thesis of Fokuoh [3] hat the same is true for *non-consecutive* patterns, i.e., that there are $2^n(1 - o(1))$ distinct non-consecutive patterns expected in a random permutation. In this talk we will indicate a proof of this conjecture.

A key combinatorial lemma, of interest in its own right, gives a bound on the probability that $2k - r$ positions, consisting of two k windows with an overlap of r, contain isomorphic patterns. It is of key value in the proof of the conjecture; the rest is "just analysis".

This work is joint with Yonah Biers-Ariel, Jane Street Capital, Verónica Borrás-Serrano, University of Puerto Rico at Mayagüez, Isabel Byrne, University of Delaware, Nathaniel Veimau, Microsoft Azure, and Yiguang Zhang, Columbia University and Google.

The second topic is related to permutations through matrix decompositions. It was presented in Moscow at the Problem Session. The Rose McCarty Conjecture [2] is that any McCarty Matrix (an $n \times n$ matrix A with positive integer entries and each of the $2n$ row and column sums equal to n), can be additively decomposed into two other matrices, B and C, such that B has row and column sumsets both equal to a permutation π on $[n]$, and C has row and column sumsets both equal to a permutation of $\{0, 1, ...n - 1\}$. The problem can also be formulated in terms of bipartite graphs.

The McCarty conjecture was formulated by Rose McCarty in 2015. In fact, the main focus of the work of Eastham, Kay, McCarty, and Spencer [2] was to reduce the upper bound on the "total acquisition number" of a diameter two graph, from $32 \ln n \ln \ln n$ (LeSaulnier et al. [4] to a substantially smaller number. They proved that, if the McCarty conjecture was true, then this upper bound could be reduced to 4. Since we prove the McCarty conjecture in this paper, we are able to report that the total acquisition number a_t of a diameter two graph is indeed at most 4. The alternative graph theoretic formulation of the McCarty Conjecture is as follows.

McCarty's Conjecture, the bipartite multigraph version: If M is a loopless bipartite multigraph on $2n$ vertices with bipartitions X and Y such that $|X| = |Y| = n$ and for each $v \in V(B)$, $\deg(v) = n$. Suppose furthermore that there exists a subgraph H of B with the same vertex set such that for every integer $k \in [n]$, there exist $x \in X, y \in Y$ such that $\deg_H(x) = \deg_H(y) = k$. Then for all diameter 2 graphs G, $a_t(G) \leq 4$.

We used probabilistic methods to solve the conjecture. This work is joint with Lybitina Koene, Virginia Tech, and Grant Shirley, ETSU.

References

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