# One-Way Bounded Permutations and the Flatorial Numbers

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*This talk is based on joint work with Brett Stevens and Aaron Williams*

We consider *one-way bounds* on a permutation $p_1 p_2 \cdots p_n$ for a fixed parameter $r \geq 0$. For example, the *low descent* bound is $p_i > p_{i+1}$ implies $p_{i+1} \leq r$ (i.e., the small value in a descent is at most $r$). The *short right* bound is $p_i^{-1} - i \leq r$ (i.e., values move at most $r$ positions to the right relative to $12 \cdots n$). No bounds are given in the opposite direction (i.e., on ascents or leftward moves). We show that eight types of permutations are counted by the *flatorial number* $\langle n,r \rangle!$ which is $n!$ for $n < r$ and $r!(r+1)^{n-r}$ for $n \geq r$.

## Introduction

Let $S_n$ be the set of all permutations of $[n] = \{1,2,\ldots,n\}$ in one-line notation. Consider the following two subsets of $S_n$ for $n = 4$ and a fixed parameter $r = 1$ explained below.

$$\underline{\mathsf{rgt}}(4,1) = \{\underline{1}234,\ 2\underline{1}34,\ 23\underline{1}4,\ 32\underline{1}4,\ 234\underline{1},\ 243\underline{1},\ 324\underline{1},\ 423\underline{1}\} \tag{1}$$

$$\overline{\mathsf{inv}}(4,1) = \{1234,\ 12\underline{4}3,\ 1\underline{3}24,\ 1\underline{34}2,\ \underline{2}134,\ \underline{2}1\underline{4}3,\ \underline{23}14,\ 2\underline{34}1\} \tag{2}$$

$$0000 \quad 0001 \quad 0010 \quad 0011 \quad 0100 \quad 0101 \quad 0110 \quad 0111$$

The first set contains permutations $p_1 p_2 \cdots p_n$ in which $p_i^{-1} > i$ implies $i \leq r$. In other words, if a value $i$ appears further to the right than it does in the identity, then $i \leq r$. Since $r = 1$ in (1), only value 1 (underlined) can appear to the right of its position in $\mathsf{id}$. The second set contains permutations whose inversion vector[1] (shown below) $v_1 v_2 \cdots v_n$ has $v_i \leq r$ for all $i$. In other words, every value is inverted with at most $r$ smaller values. Since $r = 1$ in (2), each value is inverted with at most one smaller value.

The two sets have the same cardinality; we will prove that this is true for all $n$ and $r$. More broadly, we show that eight different types of permutations are counted by the *flatorial number* $\langle n,r \rangle!$. These numbers are like factorial numbers except that terms in the product are at most $r+1$. In particular, $\langle n,0 \rangle! = 1$ and $\langle n,1 \rangle! = 2^{n-1}$ counts (1)–(2).

$$\langle n,r \rangle! = 1 \cdot 2 \cdots r \cdot (r+1)^{n-r} = \frac{n!}{(n)_{n-r}} = \begin{cases} n! & \text{if } n < r \\ r! \cdot (r+1)^{n-r} & \text{if } n \geq r \end{cases}. \tag{3}$$

None of our equivalences are obtained by reversing indices $p_n \cdots p_2 p_1$ and/or inverting values $(n - p_1 + 1)(n - p_2 + 1) \cdots (n - p_n + 1)$. These trivial modifications known as the *symmetries of the square* extend our results to $8 \cdot 4 = 32$ equinumerous types. For example, the "low rights" in (1) become "high lefts" (i.e., $p_i^{-1} < i$ implies $i \geq n - r + 1$) by reversing and inverting. Our results are summarized in Table 1. While each type is fairly elementary, we were unable to find many references in the literature. Table 2 shows that only three of the types are listed in corresponding OEIS entries. Short descents with $r = 2$ arise using classic pattern avoidance [1]: $\overline{\mathsf{dsc}}(n,2) = \mathsf{Av}_n(312,231))$. Some non-classical avoidance results also appear in the OEIS sequences [2, 3].

[1] The *inversion vector* counts the smaller inverted values. That is, $v_i = |\{j \mid j < i \text{ and } p_j^{-1} > p_i^{-1}\}|$.

| low descents | short descents | low rights | short rights | large inverts | small inverts | early lefts | short lefts |
|---|---|---|---|---|---|---|---|
| insert value | insert value | swap value | swap value | insert index | insert index | swap index | swap index |
| small labels | large labels | small labels | large labels | small labels | large labels | small labels | large labels |
| $615\overline{84}327$ | $18564\overline{73}2$ | $785162\overline{4}3$ | $53814\overline{2}76$ | $1783\overline{6}42$ | $135\overline{8}4726$ | $754\overline{8}3216$ | $15\overline{7}68243$ |
| $\underline{\mathsf{dsc}}(n,r)$ | $\overline{\mathsf{dsc}}(n,r)$ | $\underline{\mathsf{rgt}}(n,r)$ | $\overline{\mathsf{rgt}}(n,r)$ | $\underline{\mathsf{inv}}(n,r)$ | $\overline{\mathsf{inv}}(n,r)$ | $\underline{\mathsf{lft}}(n,r)$ | $\overline{\mathsf{lft}}(n,r)$ |

Table 1: Eight types of one-way bounded permutations (see Definition 2). Each is counted by the flatorial $\langle n,r \rangle!$ via a family subtree with large ($\ell - r$ to $\ell$) or small ($\ell$ and 1 to $r$) branch labels. The sample permutations satisfy their bound for $r = 4$ (or larger) but not $r = 3$. For example, $p = 615\overline{84}327 \in \underline{\mathsf{dsc}}(8,4) \setminus \underline{\mathsf{dsc}}(8,3)$ as the smaller value in each descent is $\leq 4$ but not $\leq 3$.

| $r$ | Formula ($n \geq r$) | $n = 1,2,3,\ldots$ | OEIS | short rights | short ascent | short lefts |
|---|---|---|---|---|---|---|
| 1 | $1! \cdot 2^{n-1}$ | $1,2,4,8,16,32,64,\ldots$ | A000079 | Arndt (2009) | | |
| 2 | $2! \cdot 3^{n-2}$ | $1,2,6,18,54,162,486,\ldots$ | A025192 | | Lewis (2006) | |
| 3 | $3! \cdot 4^{n-3}$ | $1,2,6,24,96,384,1536,\ldots$ | A084509 | | Knuth (2022) | |
| 4 | $4! \cdot 5^{n-4}$ | $1,2,6,24,120,600,3000,\ldots$ | A179364† | | | Hardin (2010) |
| 5 | $5! \cdot 6^{n-5}$ | $1,2,6,24,120,720,4320,\ldots$ | A179365† | | | Hardin (2010) |

Table 2: Flatorial numbers for small $r$. Three of our permutation types appear in the OEIS entries. †Several sequences match the initial terms and have another constraint (e.g., A179357).

## Family Trees of Permutations & One-Way Bounded Permutations

We consider four *family trees* of permutations. Each tree has root 1 and $S_\ell$ appears at level $\ell$. At level $\ell$, the branches are labeled with values $b \in [\ell]$. The branch $\ell$ always inserts $\ell$ as the rightmost symbol, and identity permutations are on rightmost paths. The remaining children are obtained as follows, where $b \in [\ell - 1]$ is the branch label.

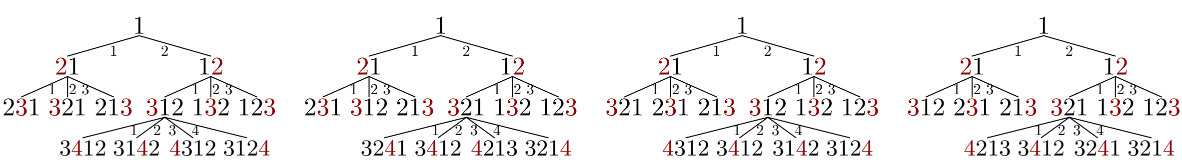
- *Insert value* tree: insert $\ell$ to the left of value $b$.
- *Swap value* tree: insert $\ell$ as the rightmost symbol, then swap it with the value $b$.
- *Insert index* tree: insert $\ell$ to the left of the index $b$.
- *Swap index* tree: insert $\ell$ as the rightmost symbol, then swap it with index $b$.

For each family tree we consider two subtrees parameterized by a given value of $r \geq 0$.

- The *large label subtree* includes branches with labels $\ell - r$ to $\ell$ at level $\ell$.
- The *small label subtree* includes branches with labels $\ell$ and 1 to $r$ at level $\ell$.

So at level $\ell$ the subtrees include branches with label $\ell$ and (at most) the $r$ smallest or largest remaining labels. Thus, the number of nodes at each successive level increases from 1 up to $r{+}1$ giving Remark 1. Family trees and subtrees appear in Figures 1–2.

*Remark* 1. A family subtree with large (or small) labels has $\langle \ell,r \rangle!$ nodes at level $\ell$.




(a) insert-value (b) swap-value (c) insert-index (d) swap-index

Figure 1: Four family trees up to level $\ell = 3$ with one node at level $\ell = 4$. A branch with label $b < \ell$ inserts $\ell$ (a) before value $b$ or (c) at index $b$, or inserts $\ell$ in the last position and then swaps it with (b) value $b$ or (d) index $b$. A branch with label $b = \ell$ inserts $\ell$ into the last position.

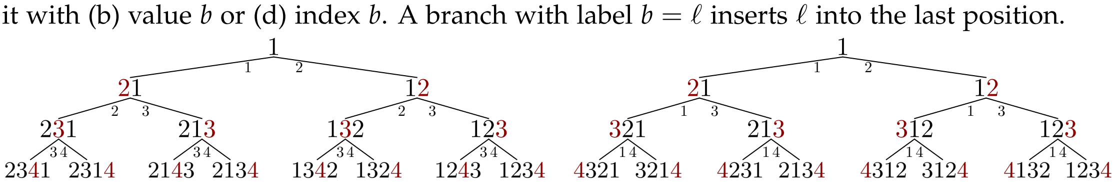



Figure 2: The insert index family subtree with large (left) and small (right) labels for $r = 1$. Note that the $\ell$ branches are included in both subtrees.

**Definition 2.** We define eight types of one-way bounded permutations for a fixed $r \geq 1$. We use the notation that $p = p_1 p_2 \cdots p_n$ is a permutation, $p_1^{-1} p_2^{-1} \cdots p_n^{-1}$ is its inverse permutation, and $v_1 v_2 \cdots v_n$ is its inversion vector. We call a value $i \in [n]$ in a permutation an *invert* if it is inverted with at least one smaller value (i.e., $v_i > 0$).

1. $p$ has *low descents*, denoted $p \in \underline{\mathsf{dsc}}(n,r)$, if $p_{i+1} < p_i$ implies $p_{i+1} \leq r$.
   That is, the smaller value in a descent is at most $r$.
2. $p$ has *short descents*, denoted $p \in \overline{\mathsf{dsc}}(n,r)$, if $p_i - p_{i+1} \leq r$.
   That is, each descent has height at most $r$.
3. $p$ has *low rights*, denoted $p \in \underline{\mathsf{rgt}}(n,r)$, if $p_i^{-1} > i$ implies $i \leq r$.
   That is, only small values can move to the right.
4. $p$ has *short rights*, denoted $p \in \overline{\mathsf{rgt}}(n,r)$, if $p_i^{-1} - i \leq r$, or equivalently, $p_i \geq i - r$.
   That is, values can move at most $r$ spaces to the right.
5. $p$ has *large inverts*, denoted $p \in \underline{\mathsf{inv}}(n,r)$, if $v_i > 0$ implies $v_i \geq i - r$.
   That is, any invert is inverted with at least $i - r$ smaller values.
6. $p$ has *small inverts*, denoted $p \in \overline{\mathsf{inv}}(n,r)$, if $v_i \leq r$ for all $i$.
   That is, each value is inverted with at most $r$ smaller values.
7. $p$ has *early lefts*, denoted $p \in \underline{\mathsf{lft}}(n,r)$, if $p_i^{-1} < i$ implies $p_i^{-1} \leq r$.
   That is, if a value is moved to the left, it must be in the first $r$ positions of $p$.
8. $p$ has *short lefts*, denoted $p \in \overline{\mathsf{lft}}(n,r)$, if $p_i^{-1} \geq i - r$, or equivalently, $p_i \leq i + r$.
   That is, values can move at most $r$ spaces to the left.

**Theorem 3.** *For all $n \geq 1$ and $r \geq 1$, each of the sets of one-way bounded permutations in Definition 2 are counted by flatorial numbers, $\langle n,r \rangle!$.*

*Proof.* By Remark 1, for each set of one-way bounded permutations, we need only show that the permutations in the set are exactly the nodes in the associated subtree with large labels or small labels. We illustrate this for low rights using the swap value tree with small labels; the other proofs are similar. A *right* is a value $v$ with $p_v^{-1} > v$.

Consider the swap value tree, in which each child of $p_1 p_2 ... p_{\ell-1}$ at branch $b$ is obtained by appending $\ell$, and then swapping $\ell$ with $b$ for $b < \ell$. Each branch $b < \ell$ creates a right: $b$ is now in position $\ell > b$. Suppose $p = p_1 p_2 ... p_{\ell-1} \in \underline{\mathsf{rgt}}(\ell - 1, r)$. The children of $p$ in $\underline{\mathsf{rgt}}(\ell - 1, r)$ are exactly those with branches $b \leq r$ (in which rights with values at most $r$ are created), and branch $\ell$ (in which no right is created). On the other hand, if $v$ is a right in permutation $p$, then $v$ will be a right in all descendants of $p$, since $v$ either remains in its same position for branches $b \neq v$, or moves further to the right at any branch $b = v$. Therefore, if $p \notin \underline{\mathsf{rgt}}(n,k)$, then none of its descendants can be. $\square$

## References


[1] David Bevan. Permutation patterns: basic definitions and notation. *arXiv preprint arXiv:1506.06673*, 2015.

[2] David Bevan, Derek Levin, Peter Nugent, Jay Pantone, Lara Pudwell, Manda Riehl, and ML Tlachac. Pattern avoidance in forests of binary shrubs. *Discrete Mathematics & Theoretical Computer Science*, 18(Permutation Patterns), 2016.

[3] Alice LL Gao and Sergey Kitaev. On partially ordered patterns of length 4 and 5 in permutations. *The Electronic Journal of Combinatorics*, 26(3), 2019.