Emerging consecutive pattern avoidance

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This talk is based on joint work with Sergey Kirgizov

Kitaev [3], along with Mansour [4, 5], presented the enumeration of classes of permutations avoiding at least two length 3 consecutive patterns. In this work, we study the *asymptotic popularity*, that is, the limit probability to find a given pattern of size 3 at random position in a random permutations in the eighteen avoidance classes from Kitaev-Mansour works. We show that, in certain cases, some of the remaining patterns disappear asymptotically. It is a quite enchanting fact. The first known occurrence of this "emerging consecutive pattern avoidance" is found in [1], where the authors proved that the consecutive pattern 321 asymptotically disappears in the class $\mathrm{Av}(123, 132, 213)$.

Table 1 summarizes our results, presenting eighteen classes, as they appear in [3]. In this table, an empty cell means that corresponding patterns are avoided *by design*, while 0 says that the respective pattern disappears asymptotically (the probability to find this pattern at a random position in a random permutation tends to 0, as the permutation size grows). The values 1/2 and 1/4 present in this table should also be understood in the asymptotic sense. There are two "N/A" for Class 4, because this class is empty for $n > 3$. Question marks indicate open problems.

For most of the classes, the study of the asymptotic popularities is quite straightforward from the structure of the permutations in the class. However, we focused our attention on two classes where this problem is significantly harder.

Avoiding 123, 132 and 321

The first class we study in details is Class 11, namely $\mathrm{Av}(123, 132, 321)$. To do so, we use an enumerative approach, computing the exact number of occurrences of each pattern in the class.

Theorem 1. *For $p \in \{213, 231, 312\}$ and $n \in \mathbb{N}$, let $\mathbf{p}_n$ denote the total number of occurrences of $\mathbf{p}$ in the permutations of $\mathrm{Av}_n(123, 132, 321)$. Then,*

$$\mathbf{231}_n = (n-1)!! \left\lceil \frac{n-3}{2} \right\rceil + (n-2)!! \left\lceil \frac{n-2}{2} \right\rceil,$$

$$\mathbf{312}_n = (n-1)!! \left(\frac{(-1)^{n-1} + n - 3}{4} + \frac{1}{2} \sum_{\substack{k=1 \\ k \neq n \mod 2}}^{n-1} \frac{1}{k} \right) + (n-2)!! \left(\frac{(-1)^{n} + n - 4}{4} + \frac{1}{2} \sum_{\substack{k=1 \\ k = n \mod 2}}^{n-2} \frac{1}{k} \right),$$

$$\mathbf{213}_n = (n-1)!! + (n-2)!! - \mathbf{231}_n - \mathbf{312}_n,$$

where $n!!$ is defined by $0!! = 1$, and for $n \geq 1$

$$n!! = \begin{cases} n \cdot (n-2) \dots 3 \cdot 1 & \text{if } n \text{ is odd,} \\ n \cdot (n-2) \dots 4 \cdot 2 & \text{if } n \text{ is even.} \end{cases}$$

Class \ Pattern	123	132	213	231	312	321
1 (simple)			1/2	1/2		
2 (simple)				0		1
3 (simple)	1/2					1/2
4 (simple)			N/A		N/A	
5 (simple)	1			0		
6 (simple)				1/2	1/2	
7 (done in [1])				1/2	1/2	0
8 (simple)			0		0	1
9 (simple)	1/2				0	1/2
10 (open)			?	?		?
11 (Theorem 1)			1/4	1/2	1/4	
12 (open)		?	?			?
13 (open)		?	?		?	?
14 (open)	?	?			?	?
15 (open)	?			?	?	?
16 (simple)		1/4	1/4	1/4	1/4	
17 (Theorem 2)			1/4	1/2	1/4	0
18 (simple)	1/2		0		0	1/2

Table 1: The asymptotic popularity patterns among eighteen avoidance classes.

The asymptotic popularities of 231, 312 and 213 can then be deduced directly from Theorem 1 (see Table 1).

Avoiding 123 **and** 132

The other class we focused our attention on is Class 17, namely $\mathrm{Av}(123, 132)$. In [2], Claesson proved that the Foata transform induces a bijection between $\mathrm{Av}_n(123, 132)$ and $\mathcal{I}_n$, the set of involutions of size n. We then take advantage of the rarity of fixed points in $\mathcal{I}_n$ to compute the asymptotic popularity of 321 and 231 (see Table 1). Table 2 shows the correspondence between patterns in Class 17 and patterns in $\mathcal{I}_n$.

Pattern in $\mathrm{Av}_n(123, 132)$	Pattern in $\mathcal{I}_n$, with $a < b < c$	Fixed point-free pattern in $\mathcal{I}_n$
321	$(c)(b)(a)$ or $(c)(b)(a\ \star)$ or $(\star\ c)(b)(a)$	$\emptyset$
231	$(b\ c)(a)$ or $(b\ c)(a\ \star)$	$(b\ c)(a\ \star)$
213	$(b)(a\ c)$ or $(\star\ b)(a\ c)$	$(\star\ b)(a\ c)$
312	$(c)(a\ b)$ or $(\star\ c)(a\ b)$	$(\star\ c)(a\ b)$

Table 2: The correspondence between patterns in $\mathrm{Av}_n(123, 132)$ and $\mathcal{I}_n$.

For 213 and 312, thanks to the Foata transform, we show that it is enough to study the popularity of the pattern 2314, which we do analytically.

Theorem 2. *Let* $G(z) = \sum_{n=4}^{+\infty} \frac{\mathbf{2314}_n}{n!} z^n$ *be the EGF of* $(\mathbf{2314}_n)_{n\geq 4}$*. Then*

$$G(z) = \frac{e^{\frac{(1+z)^2}{2}}}{2} \int_0^z e^{-\frac{(1+t)^2}{2}} dt + \frac{z(z-2)e^{z+\frac{z^2}{2}}}{4}.$$

With Theorem 2 and some complex analysis, we can then deduce the asymptotic popularity of 213 and 312 (see Table 1).

References

[1] J.-L. Baril, A. Burstein, S. Kirgizov. Pattern statistics in faro words and permutations *Discrete Mathematics*, 344(8), 112464 (2021).

[2] A. Claesson, Generalised pattern avoidance, *European Journal of Combinatorics*, 22(7), 961-971 (2001).

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[5] S. Kitaev, T. Mansour, On multi-avoidance of generalized patterns *Ars Combinatorica*, 76, 321-350 (2005).