Enumerating grid classes using MSOL

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This talk is based on joint work with Michal Opler

Let M be a $r \times c$ matrix with entries $\{-1, 0, 1\}$. A *monotone grid class* $\mathrm{Grid}(M)$ is a set of all permutations that admit a grid-like decomposition into $r \times c$ blocks such that the block in the i-th column and j-th row induces an increasing permutation if $M_{i,j} = 1$, a decreasing one if $M_{i,j} = -1$ and the block is empty if $M_{i,j} = 0$. The class $\mathrm{Grid}(M)$ is *acyclic* if there is no cycle in M, i.e., no sequence of its distinct non-zero entries $c_0, c_1, \ldots, c_{k-1}$ such that the c_i and $c_{(i+1) \bmod k}$ are either in the same column or in the same row and the whole sequence is neither in a single column nor in a single row.

Table 1: Selected results

Class	M	Generating function
L_1	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	$-\frac{2x^4 - 6x^3 + 4x^2 - x}{2x^4 - 9x^3 + 12x^2 - 6x + 1}$
L_4	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	$-\frac{x^4 - 4x^3 + 3x^2 - x}{(1-x)^2(1-3x+x^2)}$
T_5	$\begin{pmatrix} -1 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix}$	$\frac{(14x^5 - 56x^4 + 71x^3 - 39x^2 + 10x - 1)x}{(2x^2 - 4x + 1)(x^2 - 3x + 1)(3x - 1)(x - 1)^2}$
J_3	$\begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$-\frac{(3x^4 - 13x^3 + 17x^2 - 7x + 1)x}{(2x^2 - 4x + 1)(x^2 - 3x + 1)(2x - 1)}$
J_{10}	$\begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 0 \end{pmatrix}$	$-\frac{(2x^4 - 9x^3 + 13x^2 - 6x + 1)x}{(2x^2 - 4x + 1)(2x - 1)(x - 1)^2}$
Z_3	$\begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$	$-\frac{(3x^5 - 4x^4 - 15x^3 + 18x^2 - 7x + 1)x}{(x^2 - 3x + 1)(3x - 1)(2x - 1)(x - 1)}$
W_{10}	$\begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$	$-\frac{(x^5 - 7x^4 + 19x^3 - 18x^2 + 7x - 1)x}{(x^3 - 6x^2 + 5x - 1)(x^2 - 3x + 1)(x - 1)}$

Braunfeld [2] recently showed how to represent acyclic monotone grid classes[1] in monadic second order logic (MSOL). We implement a Python program which given an acyclic matrix M generates the proposed representation of $\mathrm{Grid}(M)$ in MONA language. MONA [3] is then used to generate a deterministic finite automaton A over alphabet with 2^k symbols where k is the number of non-zero entries in M (but the automaton uses only k letters of this alphabet).

An important property of this representation is that there exists a bijection between permutations in $\mathrm{Grid}(M)$ and strings accepted by A, and moreover this bijection

[1] The paper talks about geometric grid classes but we do not support general geometric grid classes yet, only acyclic ones.

is length preserving. Hence to count how many permutations of size n belong in Grid(M), it is enough to count strings of length n accepted by A. We use Sage [4] to obtain generating function counting these string by solving system of linear equations obtained from transition matrix of A.

Moreover, our implementation admits imposing additional restrictions on permutations in the class like being simple, sum- or skew-indecomposable or avoiding given pattern.

We use this program to calculate generating functions and bases of some acyclic monotone grid classes. We are able to do this for most classes with 4 and some with 5 non-zero entries obtaining new results and confirming some conjectured ones [1]. Some of them with simple generating functions are shown in Table 1. We also intend to make this program publicly available.

References

[1] David Bevan, 2024, Private communication.

[2] Samuel Braunfeld, *Decidability in geometric grid classes of permutations*, Proceedings of the American Mathematical Society **153** (2024), no. 3, 987–1000.

[3] Nils Klarlund and Anders Møller, *MONA Version 1.4 User Manual*, BRICS, Department of Computer Science, University of Aarhus, 2001.

[4] The Sage Developers, *SageMath, the Sage Mathematics Software System*, 2022, DOI 10.5281/zenodo.6259615.