The well quasi-order problem for combinatorial structures under the consecutive order

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This talk is based on joint work with Nik Ruškuc

A partially ordered set is *well quasi-ordered (wqo)* if it contains neither infinite descending sequences nor infinite antichains. This property provides a notion of 'tameness' for partially ordered sets, with those which are not wqo being comparatively 'wild'. Well quasi-order has been widely studied across combinatorics, which major theorems including Higman's Lemma [2], Kruskal's Tree Theorem [6] and the Graph Minor Theorem [4].

It can be seen that every subset of a wqo set is also wqo. On the other hand, non-wqo sets may have subsets which are wqo or non-wqo, and we will ask when these subsets are wqo? In particular, we will consider well quasi-order for subsets of partially ordered sets, called *avoidance sets*, which are defined by their forbidden substructures. Given a partially ordered set $(C, \leq)$ and $B \subseteq C$, the avoidance set of B is the set:

$$\mathrm{Av}(B) = \{x \in C : b \not\leq x \ \ \forall b \in B\}.$$

Avoidance sets give rise to natural algorithmic decidability questions: given a finite set B, we ask about decidability of properties of $\mathrm{Av}(B)$. We will focus on one such question: the *well quasi-order problem*. This asks if it is decidable, given a finite set B, whether $\mathrm{Av}(B)$ is wqo? In other words, is there an algorithmic distinction between wqo and non-wqo avoidance sets [1]?

The objects of study for this talk will be partially ordered sets (posets) of combinatorial structures in which two elements are related when one is a substructure of the other, in some sense. For example, permutations under the classical and consecutive subpermutation orders are two such posets. While the wqo problem is notably open under the classical order, progress has been made for the consecutive order, under which the wqo problem has been shown to be decidable [3]. To attain this result, McDevitt and Ruškuc encoded permutations as paths in certain factor graphs, and found conditions on factor graphs which determine wqo for avoidance sets of permutations.

This talk will explore recent generalisations of these techniques, which enable us to tackle the wqo problem for a range of structures under consecutive orders. To extend the notion of the consecutive order to combinatorial structures more generally, we relate two structures when one embeds in the other such that 'consecutive' elements remain consecutive in the image. We will begin by defining factor graphs for such posets and demonstrating how we can often encode structures as paths in these factor graphs.

We will explore the solution to the wqo problem in the cases where each path encodes more than one structure, and show how this yields decidability of the wqo problem for a wide class of structures, including graphs, digraphs, and collections of relations. Similarly, we will see that the wqo problem is decidable when each path encodes exactly one structure. These two cases provide insight into conditions forbidding wqo for the intermediate cases where paths may be associated with one or more structures, and invite investigation into these cases. We will highlight the solution to the wqo problem for two interesting examples of structures for which this occurs: permutations and equivalence relations [3], [5]. Additionally, we will discuss connections between the results presented so far and compare the various conditions on factor graphs determining wqo, showing that perhaps a spectrum of conditions is emerging.

To finish, we will present some examples of posets of structures in which intriguingly we cannot encode elements as paths in factor graphs in the way described so far. We will outline how we may solve the wqo problem for these examples using variations of the techniques discussed during the talk.

References

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