Factorial Languages with applications to $\mathcal{V}$-antichains.

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This talk is based on joint work with Robert Brignall

The existence of the infinite antichain of oscillations at growth rate $\kappa \approx 2.20557$ marks an important phase transition in the set of possible growth rates of permutation classes. This is the smallest infinite antichain, and Vatter [1] and Bevan [2] have used it to construct permutation classes of every growth rate greater than $\lambda_A \approx 2.35698$. We begin by explaining that these constructions would in fact work for *any* infinite antichain of permutations, thus motivating a systematic search for antichains as a means of understanding the finer structure of the set of permutation class growth rates. Motivated by a desire to generalise the antichain at κ using pin sequences (a construction first introduced by Brignall, Huczynska and Vatter [3] as a means of studying simple permutations), we introduce a remarkable permutation class $\mathcal{V}_c$ occuring at growth rate $\nu_c \approx 3.51205$: this is a well-quasi-ordered class (meaning it contains no infinite antichains) whose one-point extension $\mathcal{V}_c^{+1}$ nevertheless contains uncountably many distinct infinite antichains. We shall show that these so-called $\mathcal{V}$-antichains can be generated using binary sequences over $\{0,1\}$ and that their growth rates depend only on the *recurrent complexity* of the defining binary sequence (the sequence counting the number of factors that recur infinitely often). This results in extraordinarily large collections of distinct infinite antichains occuring at the same growth rates, and will allow us to identify an interesting new point of phase transition in the set of growth rates of permutation classes, at a constant $\nu_{\mathcal{L}} \approx 3.28277$. We shall show that this constant $\nu_{\mathcal{L}}$ is an accumulation point of genuinely distinct infinite antichains from both above and below, as well as demonstrating the appearance at $\nu_{\mathcal{L}}$ of uncountably many distinct *minimal* antichains. We can also demonstrate the existence of a (very thin) interval occuring close to 3.5 in which every number is the growth rate of an antichain. These results strongly suggest that a full classification of antichains beyond growth rate $\nu_{\mathcal{L}}$ may be impossible, but also motivate a plausible conjecture about the growth rates of antichains below this point.

Finally, taking the relationship between $\mathcal{V}$-antichains and binary sequences to its conclusion, we consider the study of factorial languages (that is, factor-closed sets of words over $\{0,1\}$) in its own right and introduce the *realisability problem*: that of determining whether a given factorial language can be realised as the set of recurrent factors of some binary sequence. We shall prove some partial results towards this problem and consider the consequences for a potential further classification of antichains.

References

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