Topology of Permutation Patterns

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This talk is based on ongoing joint work with Priyavrat Deshpande and Anurag Singh

Given a fixed pattern, one can associate a simplicial complex to any permutation by collecting its subsequences that avoid the pattern. Recently, researchers have begun to take interest in understanding the topology of these complexes. In this ongoing work, we investigate the relation between structural properties of the permutation (usually relating to patterns) and topological/combinatorial properties of these complexes.

Preliminaries

Given $\pi \in \mathfrak{S}_n$ and $\sigma \in \mathfrak{S}_k$, with $k \leq n$, we define a simplicial complex X_π^σ on $[n]$ as follows: a subset $\{i_0, \ldots, i_k\}$ forms a simplex if the corresponding subsequence $\pi_{i_0} \cdots \pi_{i_k}$ avoids σ. We refer the reader to Kozlov's book [2] for all the relevant terminology concerning simplicial complexes. For example, the faces of X_π^{21} correspond to the occurrences of identity permutations (of various lengths) in π. The aim of our project is to investigate the relationship between the topology of X_π^σ and the combinatorics of permutation patterns; with the hope of bringing a more structural and qualitative aspect to enumerative and discrete problems.

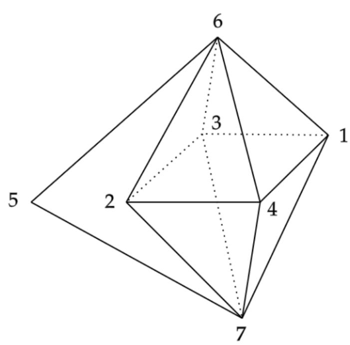

Figure 1: [3, Figure 1] showing X_π for $\pi = 3254176$. Note that $X_\pi \simeq S^1 \vee S^2$.

We will mainly focus on the case $\sigma = 21$ and mention results for other patterns σ in the last section. We denote X_π^{21} by just X_π. Now we briefly state the connection of X_π to other areas of mathematics and some recent work. Recall that the *inversion graph* G_π of $\pi \in \mathfrak{S}_n$ is the graph with vertex set $[n]$ and there is an edge between i and j if $i < j$ and $\pi(i) > \pi(j)$. The so-called *independence complex* of G_π is precisely X_π; this is because the increasing subsequences in π correspond to independent subsets of G_π.

The first appearance of X_π, in a slighlty different guise, can be traced to [4]; where the authors prove that the complex X_π is homotopy equivalent to a (disjoint union of) wedge of spheres. This is a rather surprising result. Such simplicial complexes are sought after by researchers in topological combinatorics. A reproof of this result has also appeared in [1]. In a recent article, Meshulam-Moyal [3] consider the problem of computing the probability that X_π is not topologically r-connected. This paper is one of the motivations for this project. Since X_π is a wedge of spheres, one should try and find the exact decomposition (not just the connectivity) in terms of properties related to permutation patterns.

Results

A simplicial complex is determined by its facets (maximal faces). Note that the complex X_π is not necessarily pure, *i.e.*, its facets need not have the same dimension. A facet of dimension $k-1$ corresponds to an occurrence of the mesh pattern $(12\cdots k, \{(i,i) \mid i \in [0,k]\})$. In particular, facets of dimension 0 in X_π correspond to *strong fixed points* of the reverse of π. A strong fixed point in σ is an index i such that $\sigma(j) < i$ for all $j < i$, and $\sigma(j) > i$ for all $j > i$.

A particular class of patterns plays an important role in the study of X_π. For any $k \geq 0$, the *cross-pattern of dimension k* is given by $\mathrm{cp}_k = \oplus^{k+1}21$, the direct sum of $(k+1)$ copies of 21. Note that $X_{\mathrm{cp}_k} \simeq S^k$, the k-sphere, since X_{cp_k} can be viewed as the boundary of the k-dimensional cross-polytope. The importance of these permutations comes from the following result. Recall that the homotopy dimension of a space is the minimum dimension of a cell complex homotopy equivalent to it.

Proposition 1. *For $\pi \in \mathrm{Av}(\mathrm{cp}_k)$, the homotopy dimension of X_π is at most $k-1$.*

We also characterize the permutations that maximize homotopy dimension.

Proposition 2. *For any $\pi \in \mathfrak{S}_n$, the homotopy dimension of X_π is at most $\lfloor \frac{n}{2} \rfloor - 1$. This upper bound is achieved only by $\mathrm{cp}_{\frac{n}{2}-1}$, if n is even and by precisely $n^2 - 5\binom{\frac{n-1}{2}} - 1$ permutations, if n is odd.*

One can also relate the existence of spheres in the homotopy type of X_π to patterns in π. The following result can also be stated using the language of mesh patterns.

Proposition 3. *If a permutation π contains an occurrence of the pattern cp_k such that at least one increasing subsequence in this occurrence corresponds to a facet of X_π, then the complex contains an embedded k-sphere, which is homologically non-trivial.*

We also have interesting results for special classes of permutations.

Proposition 4. *Let $\pi \in \mathrm{Av}_n(3412)$ be an involution and $\pi = \pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_k$ where each π_i is sum indecomposable.*

1. *X_π is disconnected if and only if $\pi(1) = n$ (equivalently, $k = 1$).*
2. *X_π is contractible if and only if π has a strong fixed point.*
3. *If $k \geq 2$ and π has no strong fixed points, then X_π is $(k-2)$-connected.*

4. *The complex X_π is pure if and only if each π_i is decreasing.*

We have similar results for other special classes of permutations such as Grassmannian permutations and permutations avoiding a pattern of lengths 3. For example, we have the following result.

Proposition 5. *The permutations $\pi \in \mathrm{Av}_n(132)$ such that X_π is pure correspond to Dyck paths of semilength n that have all peaks at the same height, which are counted by A007059.*

Cohen-Macaulay simplicial complexes, which lie at the crossroads of combinatorics, topology, and commutative algebra, are of interest to many researchers. One of the aims of our project is to identify, in terms of pattern information, which of these complexes are Cohen-Macaulay. In this regard, we have the following result.

Proposition 6. *If $\pi \in \mathrm{Av}(2143)$, then the complex X_π is vertex decomposable (strongest of the Cohen-Macaulay conditions).*

Future directions and open problems

1. From the results above, if $\pi \in \mathrm{Av}(2143)$ is skew indecomposable, then X_π is contractible. Similarly, if π has a strong fixed point, then X_π is contractible. Characterize π such that X_π is contractible.
2. Characterize the possible homotopy types of X_π, where π varies over permutations with k descents. We have done so for $k = 0, 1, 2$. We have also shown that, for a fixed k, there are only finitely many possible homotopy types.
3. Characterize π for which X_π is pure. These are the permutations where every increasing subsequence is part of a longest increasing subsequence.
4. Let δ_k denote the decreasing permutation of size k. We now have that for all $k \geq 3$, the complex $X_\pi^{\delta_k}$ has the homotopy type of a wedge of spheres. Investigate other properties of this complex.

References

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