Higher dimensional floorplans and Baxter d-permutations

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This talk is based on joint work with Nicolas Bonichon and Adrian Tanasa [1]

Baxter permutations and Baxter d-permutations

Baxter permutations are a central family of permutations that are in relation with numerous other combinatorial objects. They are also known to be characterised by the forbidden vincular patterns $2-41-3$ and $3-14-2$. The *separable permutations*, are also a well studied permutation class that form a subset of the Baxter permutations. A permutation is separable if it is of size 1 or if it is the skew or direct sum of two separable permutations. These permutations are additionally characterised by the forbidden patterns 2413 and 3142.

In this presentation, we are interested in higher dimensional analogs of these permutation classes. In higher dimensions, permutations are generalised by *multidimensional permutations*, also called d-permutations [3, 4]. A d-permutation of $[n]$ is a sequence of $d-1$ permutations $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_{d-1})$. Given a d-permutation $\boldsymbol{\pi}$, π_0 is the identity permutation on $[n]$. The *diagram* of a d-permutation $\boldsymbol{\pi}$ is the set of points in $P_{\boldsymbol{\pi}} := \{(\pi_0(i), \pi_1(i), \ldots, \pi_{d-1}(i)), i \in [n]\}$. Figure 2 shows (on the right) an example of 3-permutation.

Definition 1. Let $\mathbf{i} = i_1 \ldots i_{d'}$ be a sequence of indices in $\{0, \ldots, d\}$, let also $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_{d-1})$ be a d-permutation of size n. The *projection* on $\mathbf{i}$ of $\boldsymbol{\sigma}$ is the $d'-$permutation given by $\text{proj}_{\mathbf{i}}(\boldsymbol{\sigma}) := (\sigma_{i_2}\sigma_{i_1}^{-1}, \ldots, \sigma_{i_{d'}}\sigma_{i_1}^{-1})$. A projection is *direct* if $i_1 < i_2 < \cdots < i_{d'}$.

Definition 2. Let the d-permutation $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_{d-1}) \in S_n^{d-1}$ and the $d'-$permutation $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_{d'-1}) \in S_k^{d'-1}$ with $k \leq n$ and $d' \leq d$. Then $\boldsymbol{\sigma}$ *contains the pattern* $\boldsymbol{\pi}$ if there exists a direct projection $\boldsymbol{\sigma}' = \text{proj}_{i}(\boldsymbol{\sigma})$ of dimension d' and a set of indices $c_1 < \ldots < c_k$ s.t. $\sigma_i'(c_1) \ldots \sigma_i'(c_k)$ is order-isomorphic to π_i for all i. A permutation avoids a pattern if it doesn't contain it. Let s be a symmetry operation of the $[n]^d$ grid (seen as a d-cube), $s(P_{\boldsymbol{\pi}})$ is a diagram of a d-permutation that we denote $s(\boldsymbol{\pi})$. We also denote by $\text{Sym}(\boldsymbol{\pi})$ the family of permutations obtained by applying the symmetries of the d-cube on $\boldsymbol{\pi}$.

Let also p_i and p_j be the i^{th} and j^{th} points with respect to the axis 0 in a d-permutation σ. The points p_i and p_j are *k-adjacents* if one has $\sigma_k(p_i) = \sigma_k(p_j) \pm 1$. Additionally, p_i and p_j are 0-adjacent if $i = j \pm 1$.

Definition 3. A *generalised vincular pattern* $\boldsymbol{\pi}|_{X_0, \ldots, X_{d-1}}$ is a d-permutation $\boldsymbol{\pi}$ of size k along with a list of *adjacencies* given by subsets of $[k-1]$. Given a d-permutation σ, the set of points $p_1 \ldots p_k$ is an occurrence of $\boldsymbol{\pi}|_{X_0, \ldots, X_{d-1}}$ if it is an occurrence of $\boldsymbol{\pi}$ and

[1] This presentation is based on [1].

if it satisfies that for any j in any X_k the j^{th} and the $j+1^{th}$ point (of the occurrence) with respect to the axis k are k-adjacent.

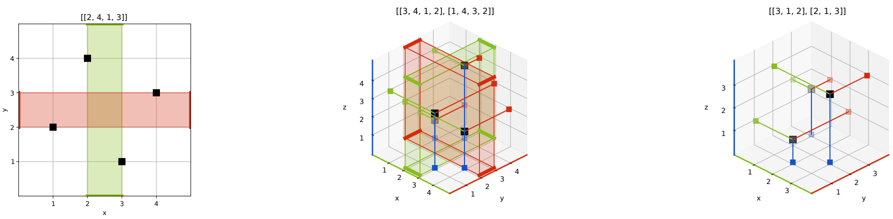

Figure 1: The patterns $2413|_{2,2}$, $(3412, 1432)|_{2,2,.}$ and $(312, 213)$.

In [3] and [4], a definition of *separable d-permutations* and of *Baxter d-permutations* was proposed. It was shown that these generalizations were characterized by the forbidden patterns $\mathrm{Sym}(2413), \mathrm{Sym}((312, 213))$ for separable d-permutations and by the forbidden patterns $\mathrm{Sym}(2413|_{2,2})$, $\mathrm{Sym}((312, 213)|_{1,2,.})$, $\mathrm{Sym}((3412, 1432)|_{2,2,.})$, $\mathrm{Sym}((2143, 1423)|_{2,2,.})$ for Baxter d-permutations.

Higher dimensional floorplans

A *floorplan* of size n is a partition of a rectangle using n interior-disjoint rectangles. These combinatorial objects have been studied in various fields of computer science, architecture or discrete geometry.

A *d-dimensional floorplan* (or a *box partition*) is a partition of a d-dimensional hyperrectangle with n interior-disjoint d-dimensional hyperrectangles (called blocks). A $(d-1)$-hyperrectangles with fixed coordinate x_i is called *facet* of axis i. Given an axis i, the boundary of a d-dimensional hyperrectangle defines two facets of such axis. A *d-dimensional floorplan* is *generic* if the set of facets that share the same i-th coordinate is a single facet. A *border* is a maximal facet of the interior of the bounding d-hyperrectangle.

Definition 4. A *d-floorplan* is a generic d-dimensional floorplan that has no border crossing.

Two d-floorplans are equivalent if the relative positions (up, down, left, right etc...) of their boxes are the same. Figure 2 (on the left) shows an example of 3-floorplan. Mosaic floorplans (or $2-$floorplans) are known to be in bijection with Baxter permutations [2]. This bijection also defines a bijection between separable permutations and a subset of mosaic floorplans, called *guillotine partitions*.

In [3], a generalization to arbitrary dimensions of guillotine partitions, was considered. The authors find a bijection between $2^{d-1}-$dimensional guillotine partitions and separable d-permutations.

In the work presented here, we introduce a generalization of the bijection [2] between mosaic floorplans and Baxter permutations which also generalizes the one in [3].

Theorem 5. *There exists a bijection betweeen d-permutations with n points avoiding* Sym((312, 213)), Sym($2413|_{2,2}$) *and* 2^{d-1}*-floorplans with n blocks.*

The multipermutations involved in the bijection are defined by the avoidance of the vincular patterns of Baxter permutations and of the dimension 3 patterns of the separable d-permutations. This set is strictly included in the set of Baxter d-permutations defined in [4]. Table 1 summarizes the different objects and related permutations classes.

Objects	Permutations	Pattern avoidance
Slicing floorplans	Separable	Sym(2413)
Mosaic floorplans	Baxter	Sym($2413\|_{2,2}$)
2^{d-1}-guillotine floorplans	d-Separable	Sym(2413), Sym((312, 213))
2^{d-1}-Floorplans	sub d-Baxter	Sym($2413\|_{2,2}$), Sym((312, 213))
	$d-$Baxter	Sym($2413\|_{2,2}$), Sym($(312, 213)\|_{1,2,.}$), Sym($(3412, 1432)\|_{2,2,.}$), Sym($(2143, 1423)\|_{2,2,.}$)

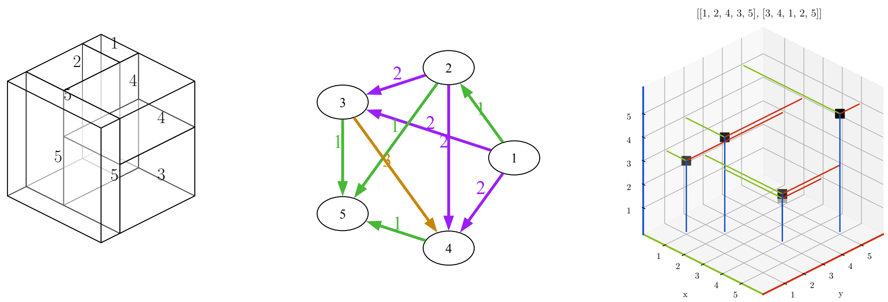

Figure 2: On the left an example of a 3−floorplan. In the middle the relative order of each blocks with respect to each direction (x, y, z). On the right the corresponding 3-permutation (considering the 3-floorplan as a 4-floorplan).

References

[1] Nicolas Bonichon and Thomas Muller and Adrian Tanasa (2025). *Higher dimensional floorplans and Baxter d-permutations*. https://arxiv.org/abs/2504.01116

[2] Ackerman, Eyal and Barequet, Gill and Pinter, Ron Y (2006). *A bijection between permutations and floorplans, and its applications*. *Discrete Applied Mathematics* 154:1674–1684

[3] Asinowski, Andrei and Mansour, Toufik (2010). *Separable d-permutations and guillotine partitions*. *Ann. Comb.*, 14:17–43.

[4] Bonichon, Nicolas and Morel, Pierre-Jean (2022). *Baxter d-Permutations and Other Pattern-Avoiding Classes*. *Journal of Integer Sequences*, 25:3.