The insertion encoding of Cayley permutations

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A *Cayley permutation* is a word $\pi \in \mathbb{N}^*$ such that every number between 1 and the maximum value of π appears at least once. Cayley permutations can be seen as a generalisation of permutations where repeated values are allowed. Definitions of pattern containment and Cayley permutation classes follow the same ideas as defined for permutations where the patterns contained are also Cayley permutations, so the Cayley permutation class $\mathrm{Av}(11)$ describes all permutations. Recently, Cayley permutation classes avoiding Cayley permutations of length 3 have been enumerated using species by Claesson, Cerbai, Ernst and Golab [1] as well as in Golab's thesis [2].

The set of Cayley permutations is in bijection with the set of ordered set partitions; the value i in the j^{th} block of the ordered set partition implies the i^{th} index of the Cayley permutation has value j. For example, the Cayley permutation 21321 corresponds to the ordered set partition $\{2,5\},\{1,4\},\{3\}$. Therefore, the set of Cayley permutations are counted by the Fubini numbers.

The goal of this work is to utilise some of the methods that have been developed for permutation classes to study Cayley permutation classes. In particular, we extend the insertion encoding of permutations introduced by Albert, Linton and Ruskuc [3] to Cayley permutations in two different ways. First, by inserting new maxima, which we will call the *vertical insertion encoding* and then by inserting new rightmost values, which we will call the *horizontal insertion encoding*. As permutations are closed with respect to the inverse symmetry, inserting a new maximum is equivalent to inserting a new rightmost value so these two methods are equivalent. For Cayley permutations this is no longer the case.

Our main result is to fully classify the Cayley permutation classes that have a regular insertion encoding, which follows a similar theory to the theory for permutations. A size $n+k$ Cayley permutation π is a *vertical juxtaposition* of the size n Cayley permutation σ and size k Cayley permutation τ if the smallest n values of π standardise to σ and the largest k values standardise to τ. For example, a vertical juxtaposition of 12432 and 221 is 16624352.

Similarly, a size $n+k$ permutation π is a *horizontal juxtaposition* of the size n permutation σ and the size k permutation τ if the size n prefix of π standardises to σ and the size k suffix of π standardises to τ. For example, 13245 is a juxtaposition of the increasing sequence 12 and the increasing sequence 123.

There are nine classes of vertical juxtapositions which are central to our theory. These arise from vertically juxtaposing strictly increasing, strictly decreasing or constant sequences. We will use $\mathcal{V}_{a,b}$ with $a,b \in \{D,I,C\}$ to denote the Cayley permutation class which is the vertical juxtaposition of a strictly increasing sequence if $a = I$, strictly decreasing sequence if $a = D$ and constant sequence if $a = C$ with a strictly increasing sequence if $b = I$, strictly decreasing sequence if $b = D$ and constant sequence if $b = C$. For example, the Cayley permutation 23141 belongs to the class $\mathcal{V}_{C,I}$. We also define the four classes of horizontal juxtapositions that come from horizontally juxtaposing strictly increasing and strictly decreasing sequences as $\mathcal{H}_{I,I}$, $\mathcal{H}_{I,D}$, $\mathcal{H}_{D,I}$, and $\mathcal{H}_{D,D}$. With these definitions we can state our main results.

Theorem 1. *A Cayley permutation class* $\mathrm{Av}(B)$ *has a regular vertical insertion encoding if and only if B contains a Cayley permutation from each of the nine classes* $\mathcal{V}_{I,I}$, $\mathcal{V}_{I,D}$, $\mathcal{V}_{I,C}$, $\mathcal{V}_{D,I}$, $\mathcal{V}_{D,D}$, $\mathcal{V}_{D,C}$, $\mathcal{V}_{C,I}$, $\mathcal{V}_{C,D}$, *and* $\mathcal{V}_{C,C}$ *of vertical juxtapositions.*

Theorem 2. *A Cayley permutation class* $\mathrm{Av}(B)$ *has a regular horizontal insertion encoding if and only if it B contains a Cayley permutation from each of the four classes* $\mathcal{H}_{I,I}$, $\mathcal{H}_{I,D}$, $\mathcal{H}_{D,I}$ *and* $\mathcal{H}_{D,D}$ *of horizontal juxtapositions.*

Checking if a Cayley permutation class has either type of regular insertion encoding can, therefore, be done using a linear time algorithm. Table 1 shows the number of Cayley permutation classes defined by avoiding size 3 patterns with vertical or horizontal insertion encoding that is regular. Every class defined by avoiding nine or more size three patterns has a regular insertion encoding of either type. Of the 8191 bases containing Cayley permutations of length 3 only, 7914 of them have a regular insertion encoding.

Size of basis	Number of classes	Regular vertical insertion encoding	Regular horizontal insertion encoding	Either
1	13	0	0	0
2	78	0	0	0
3	286	67	45	85
4	715	377	294	441
5	1287	960	846	1050
6	1716	1508	1433	1582
7	1716	1631	1607	1673
8	1287	1267	1264	1281
9	715	713	713	715

Table 1: The number of Cayley permutation classes defined by avoiding size 3 patterns that have a regular insertion encoding.

We also have a method for computing the regular vertical insertion encodings, which can be adapted to the horizontal case. Vatter [4] gave a method for constructing a DFA for the insertion encoding of slot-bounded permutation classes. The method from that paper can be extended to slot-bounded Cayley permutation classes. However, we present an alternative approach that is more efficient for permutations and can be applied to Cayley permutations. Our algorithm follows methods similar to those introduced by Albert, Bean, Claesson, Nadeau, Pantone, and Ulfarsson [5] for permutations, but adapted to Cayley permutations.

Cerbai [6] found the basis for hare pop-stack sortable Cayley permutations to be $\mathrm{Av}(231, 312, 2121)$ and left their enumeration as an open problem. Using our algorithm, we present the generating function $S(x)$ for $\text{Av}(231, 312, 2121)$ to be

$$S(x) = \frac{2x^2 - 2x^3 - x}{4x^3 - 6x^2 + 5x - 1},$$

for the sequence beginning $1, 3, 11, 41, 151, 553, 2023$ which is sequence A335793 in the Online Encyclopedia of Integer Sequences (OEIS) [8].

Cayley permutations are in bijection with ordered set partitions. The unordered set partitions are in bijection with a subset of Cayley permutation, referred in the literature as *restricted growth functions* [7]. Restricted growth functions are the Cayley permutations with the additional condition that for all values k and ℓ in the Cayley permutation, if $k < \ell$ then the first occurrence of k appears before the first occurrence of ℓ. Our algorithm can be adapted to compute the rational generating functions for pattern-avoiding restricted growth functions with either a horizontal or vertical regular insertion encoding. It should be possible to classify these regular cases in a similar way as we have done for Cayley permutations, but this is left for future work.

References

[1] A. Claesson, G. Cerbai, D. Ernst, and H. Golab (2024). *Pattern-avoiding Cayley permutations via combinatorial species*. https://arxiv.org/abs/2407.19583

[2] H. Golab (2024). *Pattern avoidance in Cayley permutations*. Master's thesis, Northern Arizona University.

[3] M. Albert, S. Linton, and N. Ruskuc (2005). *The insertion encoding of permutations*. Electronic journal of combinatorics

[4] V. Vatter (2009). *Finding regular insertion encodings for permutation classes*. Journal of Symbolic Computation

[5] M. Albert, C. Bean, A. Claesson, E. Nadeau, J. Pantone, and H. Ulfarsson (2023). *Combinatorial exploration: An algorithmic framework for enumeration*.

[6] G. Cerbai (2021). *Sorting Cayley permutations with pattern-avoiding machines*. Australasian Journal of Combinatorics

[7] V. Jelínek, T. Mansour (2008). *On pattern-avoiding partitions.*. Electronic Journal of Combinatorics

[8] N. J. A. Sloane (2025). *The On-Line Encyclopedia of Integer Sequences*. https://oeis.org