Logic of **321**-avoiding permutations and random processes

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Let us take a permutation property that can be defined by predicate logic. For instance, any pattern avoidance is such a property. We are interested in the density of the permutations in a given class that satisfy that property as the size of them goes to infinity. If there exists a limit for the density for any property, then we say that the class of permutations is first-order stable. We show that Av(**321**) is first-order stable.

Given $\sigma \in S_n$, we define two binary relations on $\{1, 2, \ldots, n\}$, (1) position $(<_P)$ and (2) value $(<_V)$ as follows:

$$i <_P j \text{ if and only if } i < j,$$
$$i <_V j \text{ if and only if } \sigma(i) < \sigma(j).$$

So, any permutation in S_n can be viewed as a structure with domain $A = \{1, 2, \ldots, n\}$ and a set of binary relations $R = \{<_P, <_V\}$. Then, for example, we can express the **321**-pattern avoidance as

$$\varphi_1 = \neg \left[\exists x \exists y \exists z [(x <_P y) \wedge (y <_P z) \wedge (y <_V x) \wedge (z <_V y)]\right],$$

and we say $\sigma \vDash \varphi_1$ if and only if $\sigma \in \mathrm{Av}(\mathbf{321})$. Since we are interested in the limiting density, defining $\mathrm{P}(A)$ as the probability of an event A, we have

$$\mathrm{P}(\sigma_n \vDash \varphi_1) = \frac{\mathrm{C}_n}{n!}$$

where σ_n is a randomly chosen permutation in S_n and C_n is the nth Catalan number. A second example is the sentence that "there exists an inversion", which can be symbolized as

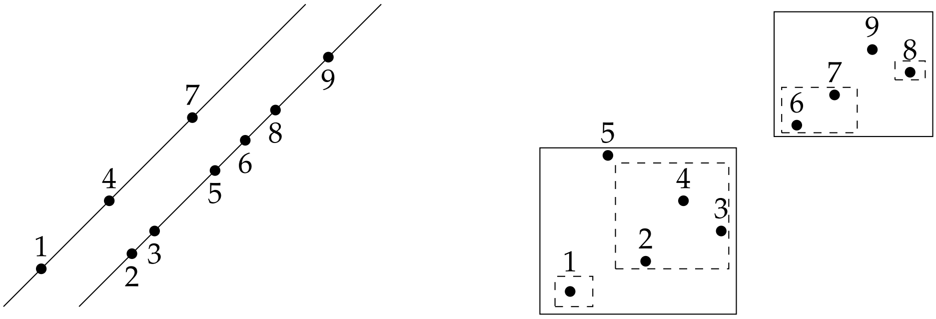
$$\varphi_2 = \exists x \exists y [(x <_P y) \wedge (y <_V x)].$$

Our main result reads

Theorem 1. [4] *Let $\sigma_n \in \mathrm{Av}_n(\mathbf{321})$ and φ be a first-order sentence on permutations. Then*

$$\lim_{n\to\infty} \mathrm{P}(\sigma_n \vDash \varphi) \text{ exists.}$$

The limit law for Av(**231**) is proven in [1] using the recursive structure of **231**-avoiding permutations. The structural differences between Av(**231**) and Av(**321**) do not allow their methods to be extended to the latter case. We, on the other hand, use random processes to exploit the representability of permutations in Av(**321**) by two increasing subsequences. See the figure below.



$\pi = \mathbf{142375689} \in \mathrm{Av}(\mathbf{321})$ $\pi = \mathbf{152436798} \in \mathrm{Av}(\mathbf{231})$

Figure 1: A pair of increasing subsequences vs. a recursive pattern

Random processes started to be used in this context very recently only with a few exceptions. In addition to the permutation classes, the logic of random graphs and random groups can be studied with analogous methods. Out of necessity, our method differs from the earlier ones, two of which are mentioned in the table below. It relies on an operator view which can be viewed as an abstraction of the processes governed by Markov chains.

Random process	Finite Markov chains	Countable Markov chains	Symbolic chains	Operator viewpoint
Example	Layered permutations [2]	Mallows random permutations [3]	No example studied	$\mathrm{Av}(\mathbf{321})$ [4]
Conditions on the matrix (operator)	stochastic irreducible aperiodic	stochastic irreducible aperiodic pos. recurrent	non-negative irreducible aperiodic pos. recurrent	(connected) (non-partite) (compact)

Table 1: Applications of random processes to show first-order stability of some permutation classes

References

[1] A. Albert, M. Bouvel, V. Féray, and M. Noy (2024). *Convergence law for 231-avoiding permutations*. Discrete mathematics and theoretical computer science, **26(1)**

[2] S. Braunfeld and M. Kukla (2022). *Logical Limit Laws for Layered Permutations and Related Structures*. Enumerative combinatorics and applications, **2:4**

[3] T. Muller, F. Skerman, T. W. Verstraaten (2023). *Logical limit laws for Mallows random permutations* https://arxiv.org/abs/2302.10148

[4] A. Özdemir (2023). *First-order convergence for* 321*-avoiding permutations* https://arxiv.org/abs/2312.01749