Characterization of Avoidance of One-Sided 2- and 3- Segment Patterns in Rectangulations by Mesh Patterns

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This talk is based on joint work with Andrei Asinowski, Torsten Mütze, and Namrata

Pattern avoidance is a fundamental concept in combinatorics, and in this work, we consider pattern avoidance in rectangulations. A *rectangulation* $\mathcal{R}$ is a decomposition of a rectangle R into a finite number of rectangles such that there are no $+$ joints. The *size* of a rectangulation is the number of rectangles in the decomposition; it is denoted by n. A *segment* of a rectangulation is a maximal straight line that consists of one or several sides of rectangles of $\mathcal{R}$, and is not included in one of the sides of R.

In order to consider equivalence classes of rectangulations, we introduce *left–right* and *above–below* relations:

- Rectangle r is *left of* rectangle r' (equivalently, r' is *right of* r) if there is a sequence of rectangles, $r = r_1, r_2, \ldots, r_k = r'$ such that the right side of r_i and the left side of r_{i+1} lie in the same segment for $i = 1, 2, \ldots, k-1$.
- Rectangle r is *below* rectangle r' (equivalently, r' is *above* r) if there is a sequence of rectangles, $r = r_1, r_2, \ldots, r_k = r'$ such that the upper side of r_i and the bottom side of r_{i+1} lie in the same segment for $i = 1, 2, \ldots, k-1$.

NW-SE labeling We label the rectangles from 1 to n according to the NW-SE order described in [3]. The rectangle with label j ($1 \leq j \leq n$) is denoted by r_j. The rectangle r_1 contains the NW (top-left) corner of R and r_n contains the SE (bottom-right) corner of R. If j is fixed, then r_i with $i < j$ are precisely the rectangles above or to the left of r_j, and r_i with $i > j$ are precisely the rectangles below or the right of r_j. For the detailed algorithm of NW-SE labeling of rectangles in a rectangulation, see [3].

Weak and strong equivalences There are two natural ways to define combinatorial equivalence for rectangulations: *weak equivalence* which preserves left-right and above-below relations between rectangles, and *strong equivalence* which also preserves contact between rectangles. Equivalence classes are called *weak rectangulations* and *strong rectangulations* and the set of all rectangulations of size n is denoted by R_n^w or R_n^s, respectively. Each weak rectangulation has a representative in which each rectangle meets the NW-SE diagonal, this representative is called a *diagonal rectangulation*. See Figure 1 for an example.

Representing rectangulations as posets and permutations Given any rectangulation $\mathcal{R}$, we can encode the contact between rectangles by a poset, called the adjacency poset and denoted $P_a(\mathcal{R})$. When the rectangulation is diagonal, we call it the weak poset and denote it $P_w(\mathcal{R})$. This is the weak poset for the entire weak rectangulation equivalence class. This poset was introduced by Law and Reading in [4], where they also described a mapping γ_w from the set of permutations S_n to the set of weak rectangulations R_n^w. In [3], Asinowski et al. generalized the adjacency poset to strong

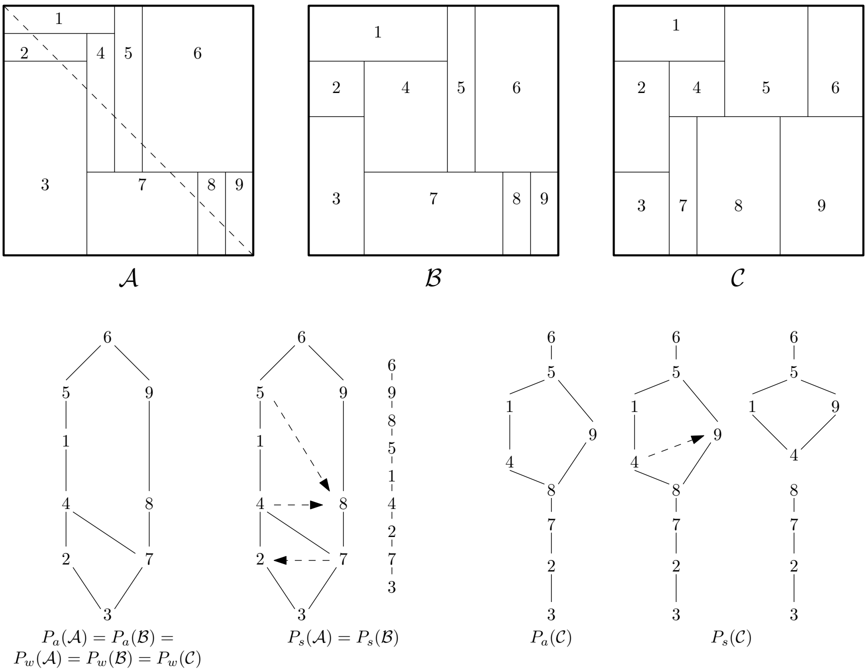
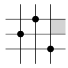
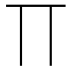
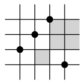
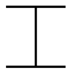
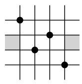
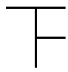
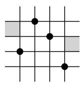
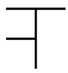
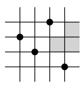


Figure 1: $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are all weakly equivalent, only $\mathcal{A}$ and $\mathcal{B}$ are strongly equivalent. $\mathcal{A}$ is a diagonal rectangulation. The adjacency poset for $\mathcal{A}$ and $\mathcal{B}$, and the weak poset for all of three is the same. The special relations added to form the strong poset from the adjacency poset are shown with dashed arrows.

rectangulations by also considering certain "special relations" when constructing the poset. That is, they begin with the adjacency poset for a strong rectangulation $\mathcal{R}$, and then add in additional relations based on certain rules. They also described a map γ_s from S_n to R_n^s. Finally, they proved that $\gamma_w^{-1}(\mathcal{R}) = \mathcal{L}(P_w(\mathcal{R}))$ and $\gamma_s^{-1}(\mathcal{R}) = \mathcal{L}(P_s(\mathcal{R}))$ where $\mathcal{L}(P)$ denotes the set of linear extensions of a poset P.

Pattern avoidance in rectangulations A rectangulation $\mathcal{R}$ contains a pattern p if there is an injection from the segments of (any representative of) p into the segments of (any representative of) R which preserves incidences, orientations, and order of incidence of neighbors of the segments of p. Otherwise, $\mathcal{R}$ avoids p. Following recent work, there has been growing interest in studying pattern avoidance in rectangulations, as they are highly versatile objects that can have connections to many other combinatorial structures [1, 2, 3]. In our work, we restrict to rectangulation patterns with two or three segments in which no segment has neighbors on both sides ("one-sided"). We give one-to-one mappings between rectangulation patterns and mesh permutation patterns such that a rectangulation $\mathcal{R}$ avoids the given rectangulation pattern if and only if the linear extensions of its poset $\mathcal{L}(P(\mathcal{R}))$ avoid the corresponding permutation patterns. The question of representing rectangulation patterns by permutation patterns was given as an interesting direction for future research by Merino and Mütze [5]. Our results are summarized for all one-sided 2- and 3-segment patterns in the table.

Rectangulation Pattern (p)	Permutation Mesh Pattern (τ_p)	weak/ strong	Enumeration for $n = 1, \ldots, 10$	OEIS
		weak	$1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796$	A000108
		strong	$1, 2, 5, 15, 51, 189, 746, 3091, 13311, 59146$	A279555
		weak	$1, 2, 6, 21, 81, 334, 1445, 6485, 29954, 141609$	
		strong	$1, 2, 6, 23, 103, 514, 2785, 16097, 98030, 623323$	
		weak	$1, 2, 6, 21, 79, 309, 1237, 5026, 20626, 85242$	A026737 & A111279
		strong	$1, 2, 6, 23, 101, 482, 2433, 12787, 69270, 384134$	
		weak	$1, 2, 6, 21, 80, 322, 1347, 5798, 25512, 114236$	A106228
		strong	$1, 2, 6, 23, 102, 499, 2622, 14547, 84229, 504775$	
		weak	$1, 2, 6, 21, 80, 322, 1347, 5798, 25512, 114236$	A106228
		strong	$1, 2, 6, 23, 102, 499, 2622, 14547, 84229, 504775$	

Our results For each rectangulation pattern p and associated permutation mesh pattern τ_p given in the table, we prove the following statement:

A permutation $\pi \in S_n$ avoids τ_p if and only if $\gamma_s(\pi)$ avoids p.

For each result we follow the same proof outline: we begin by proving that $\pi \in S_n$ avoids τ_p if and only if $\gamma_w(\pi)$ avoids p. Hence, given a weak rectangulation $\mathcal{R}$ which avoids p, every linear extension of $P_w(\mathcal{R})$ avoids τ_p. As $P_s(\mathcal{R})$ contains all of the relations included in $P_w(\mathcal{R})$, $\mathcal{L}(P_s(\mathcal{R})) \subseteq \mathcal{L}(P_w(\mathcal{R}))$, so we conclude the desired result.

References

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