Robinson–Schensted Shapes Arising from Cycle Decompositions

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This talk is based on joint work with Martha Du Preez, William Erickson, Jonathan Feigert, Markus Hunziker, Jonathan Meddaugh, Mitchell Minyard, and Kyle Rosengartner

Statement of the Problem. At the heart of classical algebraic combinatorics is the representation theory of the symmetric group S_n. In turn, much of this theory can be expressed in terms of integer partitions. In this work, we describe the subtle relationship between two partitions closely associated with each element $\sigma \in S_n$: the *cycle type* of σ, on one hand, and the *shape* of σ via the Robinson–Schensted correspondence on the other hand. Although separately each of these partitions is fundamental to the general theory, the two had not yet been studied *together* until a very recent paper [1] treating the special case where σ is a cyclic (or almost cyclic) permutation. (The most closely related works study cycle types and descents [2], or shapes and inversions [3].) A natural question is the following: which shapes arise from the elements of a given cycle type?

It is well known that the conjugacy classes of S_n (and also its irreducible complex representations) can be naturally labeled by the integer partitions α of n, written as $\alpha \vdash n$. In particular, the conjugacy class of $\sigma \in S_n$ is labeled by the partition $\alpha = (\alpha_1, \ldots, \alpha_r)$ giving the *cycle type* of σ, which is easily read off from the expression of σ in disjoint cycle notation: $\sigma = (\alpha_1\text{-cycle})(\alpha_2\text{-cycle}) \cdots (\alpha_r\text{-cycle})$. Throughout the paper, we write $\mathcal{C}_\alpha$ to denote the conjugacy class of S_n consisting of elements with cycle type α.

Another key concept in the representation theory of S_n and in algebraic combinatorics in general is the Robinson–Schensted (RS) correspondence. The RS correspondence is a bijection

$$S_n \xrightarrow{\text{RS}} \coprod_{\lambda \vdash n} \mathrm{SYT}(\lambda) \times \mathrm{SYT}(\lambda),$$

where $\mathrm{SYT}(\lambda)$ denotes the set of standard Young tableaux with shape λ, meaning that the partition λ gives the row lengths of the tableaux. If the RS correspondence takes σ to a pair $(P, Q) \in \mathrm{SYT}(\lambda) \times \mathrm{SYT}(\lambda)$, then we say that λ is the *RS shape* of σ, which we denote by writing $\mathrm{sh}(\sigma) = \lambda$. Thus in the example

$$\sigma = (3,5,4,7)(1,2,6) \xrightarrow{\text{RS}} \left(\begin{array}{|c|c|c|} \hline 1 & 3 & 7 \\ \hline 2 & 4 \\ \cline{1-2} 5 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array}\ , \ \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 7 \\ \cline{1-2} 5 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array} \right), \tag{1}$$

we have $\sigma \in \mathcal{C}_{(4,3)}$, and $\mathrm{sh}(\sigma) = (3,2,1,1)$. The main problem of our work is to describe the elements of

$$\mathcal{S}_\alpha := \{\mathrm{sh}(\sigma) : \sigma \in \mathcal{C}_\alpha\}.$$

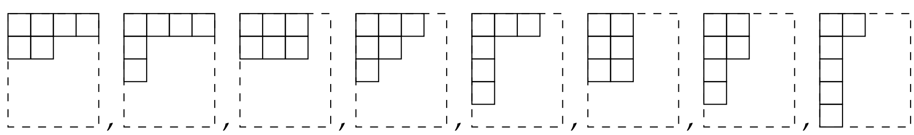
Main Result. As a preliminary result, for all $\alpha = (\alpha_1, \ldots, \alpha_r) \vdash n$, we prove that the partitions in $\mathcal{S}_\alpha$ have Young diagrams fitting inside a certain bounding box:

$$\mathcal{S}_\alpha \subseteq \mathcal{B}_\alpha, \tag{2}$$

where

$$\mathcal{B}_\alpha := \left\{ \lambda \vdash n : \begin{array}{l} \lambda \text{ has at most} \\ \big(n - r + \#\{i : \alpha_i = 2\} + \delta_{1,\alpha_r}\big) \text{ many rows and} \\ \big(n - r + \#\{i : \alpha_i = 1\}\big) \text{ many columns} \end{array} \right\}. \tag{3}$$

For example, if $\alpha = (4, 2)$, then $\mathcal{B}_\alpha$ consists of all partitions of $n = 6$ whose Young diagram fits inside the box of dimensions $(6 - 2 + 1 + 0) \times (6 - 2 + 0) = 5 \times 4$. Concretely, we have

$\mathcal{B}_{(4,2)} = \Bigg\{$  $\Bigg\}.$

For certain cycle types α, the containment in (2) is, in fact, an equality. We can thus reframe our main problem as follows: classify the cycle types α such that $\mathcal{S}_\alpha = \mathcal{B}_\alpha$, and for the remaining cycle types α, determine the complement $\mathcal{B}_\alpha \setminus \mathcal{S}_\alpha$. The following theorem is the main result of our paper, where we solve the problem in the case $r = 2$, that is, where $\alpha = (\alpha_1, \alpha_2)$.

Theorem 1. *Let n be a positive integer, and let $\alpha = (\alpha_1, \alpha_2) \vdash n$.*

1. *If n is odd, then $\mathcal{S}_\alpha = \mathcal{B}_\alpha$.*
2. *If n is even, then $\mathcal{S}_\alpha = \mathcal{B}_\alpha$ unless α occurs in the following table:*

α	$\mathcal{B}_\alpha \setminus \mathcal{S}_\alpha$
$(n-1,\ 1)$	$\{(\frac{n}{2}, \frac{n}{2})\}$
$(\frac{n}{2}, \frac{n}{2})$, where $4 \mid n$	$\{(n-2, 1, 1),\ (3, 1, \ldots, 1)\}$
$(\frac{n}{2}, \frac{n}{2})$, where $4 \nmid n$	$\{(n-2, 1, 1)\}$
$(4, 2)$	$\{(2, 2, 2)\}$
$(5, 3)$	$\{(2, 2, 2, 2)\}$

Our Theorem 1 generalizes the main result of [1], which can be restated as follows, using the language of the present paper:

1. If n is odd, then $\mathcal{S}_{(n)} = \mathcal{B}_{(n)}$ and $\mathcal{S}_{(n-1,1)} = \mathcal{B}_{(n-1,1)}$.
2. If n is even, then $\mathcal{S}_{(n)} = \mathcal{B}_{(n)}$ and $\mathcal{B}_{(n-1,1)} \setminus \mathcal{S}_{(n-1,1)} = \{(\frac{n}{2}, \frac{n}{2})\}$.

We emphasize that by introducing the bounding box $\mathcal{B}_\alpha$, we are able to state these results in a uniform manner, thus avoiding the need to designate certain "trivial shapes" and to make exceptions for small values of n.

Admissible Tableaux and α-colorings. In order to prove Theorem 1, we introduce two key combinatorial objects, which we call *admissible tableaux* and *α-colorings*. We say a standard tableau is *admissible* if it remains standard when justified along the bottom (i.e., when acted on by "gravity"). It turns out that the admissibility of a tableau Q is equivalent to the following: for any tableau P of the same shape, the permutation $\mathrm{RS}^{-1}(P,Q)$ can be read off from P by following the order of the entries in the vertical reflection of Q, which we denote by $Q^\uparrow$. In turn, for an admissible $Q \in \mathrm{SYT}(\lambda)$, an *$\alpha$-coloring* of $Q^\uparrow$ is a coloring of its entries which, via a certain canonical spiral construction, produces a permutation $\sigma \in \mathcal{C}_\alpha$ with $\mathrm{sh}(\sigma) = \lambda$. This σ cyclically permutes the entries of each color in $Q^\uparrow$ to obtain a standard P such that $\sigma = \mathrm{RS}^{-1}(P,Q)$. The crux of the paper is therefore where we construct α-colorings for all shapes λ, with the exception of the pairs (α, λ) given above in the table in Theorem 1. (There is also one family (α, λ) such that $\lambda \in \mathcal{S}_\alpha$ but no α-coloring exists; in this case we exhibit the requisite σ directly.)

Open Problems and Conjectures. As a first step toward extending the results in this paper to generic cycle types $(\alpha_1, \ldots, \alpha_r)$ where $r > 2$, we point out a special case, following from a result of Schützenberger relating to involutions, in which we can describe $\mathcal{S}_\alpha$ explicitly, namely when $\alpha_1 \leq 2$:

$$\mathcal{S}_{(2^{r-k},1^k)} = \{\lambda \vdash n : \lambda \text{ has exactly } k \text{ many columns of odd length}\}.$$

Outside of this special case described above, however, for $r > 2$, an explicit and comprehensive description of $\mathcal{S}_\alpha$ quickly becomes quite complicated. Somewhat surprisingly, the source of these complications lies entirely in the presence of repeated values among the α_i's. In fact, we conjecture that for $r > 2$, we have $\mathcal{S}_\alpha = \mathcal{B}_\alpha$ whenever $\alpha_1 > \cdots > \alpha_r > 1$. (It would then follow that for a fixed r, as $n \to \infty$, the proportion of cycle types satisfying $\mathcal{S}_\alpha = \mathcal{B}_\alpha$ approaches 100%.) See the last section of our paper for additional details, along with further conjectures and open problems concerning cycle types and α-colorings.

References

[1] A. Goel and S. Rubinstein-Salzedo (2024). *RS-complete cycle types*. *Australas. J. Combin.*, 89, 215–233.

[2] I. Gessel and C. Reutenauer (1993). *Counting permutations with given cycle structure and descent set*. *J. Combin. Theory Ser. A*, 64(2), 189-215.

[3] A. Ayyer and N. Banerjee (2022). *The number of inversions of permutations with fixed shape*. *Enumer. Comb. Appl.*, 2(4), 14p.