Extensions to Wilf-Equivalences among Partial Shuffles.

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Introduction

In [1], Bloom and Sagan introduce a *partial shuffle* as a tool for proving pattern avoidance results in the context of quasisymmetric functions. They show that taking this partial shuffle as a basis generates an infinite family of Wilf-equivalent permutation classes.

Our work expands on this result, first with a reinterpretation of the proof of Wilf-equivalence for these partial shuffle classes. We then show that Wilf-equivalence is preserved upon including a descending permutation of arbitrary length in the partial shuffle basis, giving a wider family of Wilf-equivalent bases. We then provide some enumerative results, including demonstrating the appearance of the Catalan numbers in the leading coefficient of the enumerating polynomial for these classes.

Results

Let $[\ell] = \{1, 2, \ldots, \ell\}$, and for fixed ℓ, choose $a \in [\ell]$ and b such that $a + b = \ell$. Then, the partial shuffle $\Pi(a, b)$ consists of all permutations of $[\ell]$ where every element from $[\ell] - \{a\}$ is in increasing order, apart from the strictly increasing permutation $12 \ldots \ell$.

Example 1.

$$\Pi(2, 3) = \{13452,\ 13425,\ 13245,\ 21345\}.$$

The following is Lemma 4.4 in [1]:

Lemma 2. *For pairs of natural numbers (a, b) and (c, d) such that $a + b = c + d$, the pattern avoidance classes $Av(\Pi(a, b))$ and $Av(\Pi(c, d))$ are Wilf-equivalent.*

Our proof is visual and involves defining a function S which acts on $\pi \in Av(\Pi(a, b))$, rotating certain subsets of elements to produce, after finitely many iterations, a permutation $S(\pi) \in Av(\Pi(a - 1, b + 1))$. We show that this function is injective, and that it can be reversed.

Letting δ_k denote the descending permutation of length k, we prove the following:

Theorem 3. *If δ_k is the longest descending subpermutation of $\pi \in Av(\Pi(a, b))$, then δ_k is the longest descending subpermutation in $S(\pi)$.*

An immediate corollary is that if $a + b = c + d$, then for any $k \geq 1$, $\Pi(a, b) \cup \delta_k$ is Wilf-equivalent to $\Pi(c, d) \cup \delta_k$. We determine the approximate growth of these classes:

Theorem 4. *For a, b such that $a+b=\ell$, the permutations of length n in the pattern avoidance class $Av(\Pi(a,b) \cup \delta_k)$ are counted by a polynomial in n, with degree $(\ell-2)(k-2)$.*

Proposition 5. *For a, b such that $a+b=\ell$, and $\ell \geqslant 3$, the polynomial $p(n)$ which counts $\#Av_n(\Pi(a,b) \cup \delta_3)$ has leading term $C_{\ell-2}\binom{n}{\ell-2}$, where $C_{\ell-2}$ is the $(\ell-2)^{th}$ Catalan number.*

There does not appear to be a nice closed form for the other terms in the counting polynomials, in general, although automatic enumeration schemes [2] do exist for polynomial classes such as these. With the basis $\Pi(a,b) \cup \delta_3$, we conjecture that:

Conjecture 6. *For fixed a, b, where $\ell = a+b \geqslant 3$, and $n \geq 2(\ell-2)+1$,*

$$\#Av_n(\Pi(a,b) \cup \delta_3) = C_{\ell-2}\binom{n}{\ell-2} - \sum_{1 \leq h < n-2} T_{\ell-2,h}\binom{n}{\ell-3-h},$$

where the coefficients $T_{\ell-2,h}$ correspond to rows of the transposed Catalan triangle;

$$T_{p,q} = \frac{q\binom{2p-q}{p}}{2p-q}$$

(OEIS sequence A033184).

Example 7. For $n \geq 13$, and $a+b=9$, $\#Av_n(\Pi(a,b) \cup \delta_3)$ is given by

$$429\binom{n}{7} - 132\binom{n}{5} - 132\binom{n}{4} - 90\binom{n}{3} - 48\binom{n}{2} - 20\binom{n}{1} - 6\binom{n}{0}$$

References

[1] Bloom, J. S., & Sagan, B. E. (2020). Revisiting pattern avoidance and quasisymmetric functions. *Annals of Combinatorics*, 24(2), 337-361.

[2] Homberger, C., & Vatter, V. (2016). On the effective and automatic enumeration of polynomial permutation classes. *Journal of Symbolic Computation*, 76, 84-96.