Joint Packing Densities and the Great Limit Shape

Walter Stromquist Bryn Mawr College

This talk includes results due to Sergi Elizalde, Anna de Mier, and Marc Noy (jointly), Lara Pudwell, and Miles Jones.

In this talk I'll summarize what is known about the "Great Limit Shape" (as I have been calling it), and about the "Small Limit Shape," which is the analogous construction for layered permutations. I'll consolidate results presented last fall at the Albany AMS meeting with earlier results by Sergi Elizalde and his team, some experimental observations by Lara Pudwell and Miles Jones, and some new results.

The subject is joint packing densities. If $\sigma \in \mathcal{S}_k$ (a "pattern") and $\pi \in \mathcal{S}_n$, then the *packing density* of σ in π is

$$\delta_\sigma(\pi) = (\text{ the number occurences of } \sigma \text{ in } \pi) \ / \binom{n}{k}.$$

In this talk, the patterns π always have length 3.

Given a set of patterns—say, 123 and 321—we can form their joint packing density, or *packing vector,*

$$(\ \delta_{123}(\sigma),\ \delta_{321}(\sigma)\ ) \in \mathbb{R}^2$$

for each $\sigma \in \mathcal{S}_n$. We would like to know what vectors can arise in this way.

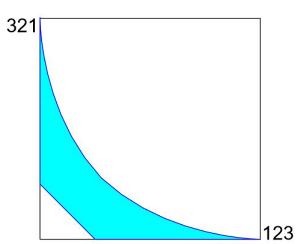
Actually, we only care about the limit for large σ. So we define the *limit shape*, $\Pi(123, 321)$, as the set of vectors in $\mathbb{R}^2$ that can occur as the limit of vectors

$$(\ \delta_{123}(\sigma_i),\ \delta_{321}(\sigma_i)\ )$$

for a sequence $\{\sigma_1, \sigma_2, \ldots\}$ of permutations of increasing size. The limit shape is a compact subset of $\mathbb{R}^2$.

(The limit shapes can also be understood in terms of permutons. A vector is in the limit shape if and only if it can occur as $(\delta_{123}(P), \delta_{321}(P)$ for some permuton P.)

For the patterns 123 and 321, the limit shape is known exactly. It was found by Sergi Elizalde and his collaborators and proved by connecting it to an analogous theorem for graphs.



$\Pi(123, 321)$ (shown as a subset of the unit square $[0, 1]^2$)

We would really like to understand the mother of all limit shapes, the Great Limit Shape:

$$\Pi^* = \Pi(123, 132, 213, 231, 312, 321).$$

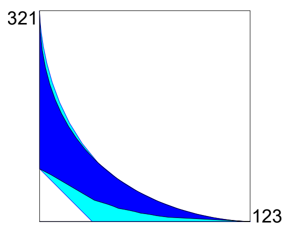
This is a subset of $\mathbb{R}^6$, but because the six packing densities add to 1, the set Π^* is contained in a 5-dimensional subspace of $\mathbb{R}^6$. A full description is still out of reach.

For now, we will have to settle for various projections and cross-sections of Π^*. We have seen $\Pi(123, 321)$, which is a projection of Π^* onto a subspace of dimension 2. Another projection of dimension 2 is $\Pi(132, 321)$, which was partially found by Sergi's team. Both Lara Pudwell and Miles Jones found evidence to identify the permutons that define the boundaries of $\Pi(132, 321)$, and now we can describe more of the picture, along with a complete description of $\Pi(132, 21)$.

Layered permutations. We can come close to a full solution of the analogous problem for layered permutations. Consider, first, the set $\Pi^L(123, 321)$, which consists of limits of packing vectors

$$(\delta_{123}(\sigma), \delta_{321}(\sigma))$$

where σ is limited to layered permutations. Some of the vectors that occur for general permutations are impossible for layered permutations.



$\Pi^L(123, 321)$ in dark color, in front of $\Pi(123, 321)$ in light color

The full limiting shape for layered permutations (the "Small Limit Shape") consists of limits of vectors of the form

$$(\delta_{123}(\sigma), \delta_{132}(\sigma), \delta_{213}(\sigma), \delta_{321}(\sigma))$$

for layered permutations σ. (We omit the terms for 213 and 312 because these patterns do not occur in layered permutations.) Because entries add to 1, these vectors form a 3-dimensional subset of $\mathbb{R}^4$. It is a cross-section of the Great Limit Shape. Here are two images of the Small Limit Set.