Inversion sequences avoiding 021 and other patterns

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In this talk, we present an ongoing work aiming at systematically solving the enumeration of inversion sequences (and ascent sequences, and restricted growth functions) avoiding 021 and any finite set of patterns.

Context

In [2], an algorithm was developed to automatically find succession rules for generating trees associated with some classes of pattern-avoiding inversion sequences. In particular, the authors used this method to compute the generating function of inversion sequences avoiding the pattern 021 and any pattern of length four or five [4, 5]. Their findings lead to some conjectures about inversion sequences avoiding 021 and another pattern; notably, they appear to always have an algebraic generating function.

Summary of results

We confirm the above conjecture, proving that inversion sequences avoiding 021 and any finite set of patterns have an algebraic generating function. We show that inversion sequences avoiding 021 and any finite set of patterns are in bijection with a language of pattern-avoiding coloured Dyck paths, which is recognized by a deterministic counter automaton. Classical results from formal language theory imply that such a language always has an algebraic generating function. Our algebraicity results also apply to ascent sequences and restricted growth functions avoiding 021 and other patterns.

Inversion sequences avoiding 021, and lattice paths

An *inversion sequence* of *length* n is a sequence of integers $\sigma = (\sigma_1, \dots, \sigma_n)$ such that $\sigma_i \in \{0, \dots, i-1\}$ for all $i \in \{1, \dots, n\}$. We also consider two subsets of inversion sequences whose values are bounded by different statistics, namely *ascent sequences* and *restricted growth functions*.

An inversion sequence avoids the pattern 10 if and only if it is nondecreasing. There is a simple bijection between 10-avoiding inversion sequences of length n and Dyck paths of length $2n$, see Figure 1 (left).

An inversion sequence avoids the pattern 021 if and only if its positive entries are nondecreasing. Indeed, since the sequence always begins with a 0, any pair of positive entries in decreasing order would complete an occurrence of 021. The bijection between 10-avoiding inversion sequences and Dyck paths can be extended to a bijection between 021-avoiding inversion sequences and a family of coloured Dyck paths.

This bijection first appeared in [3]. See Figure 1 (right) for an illustration.

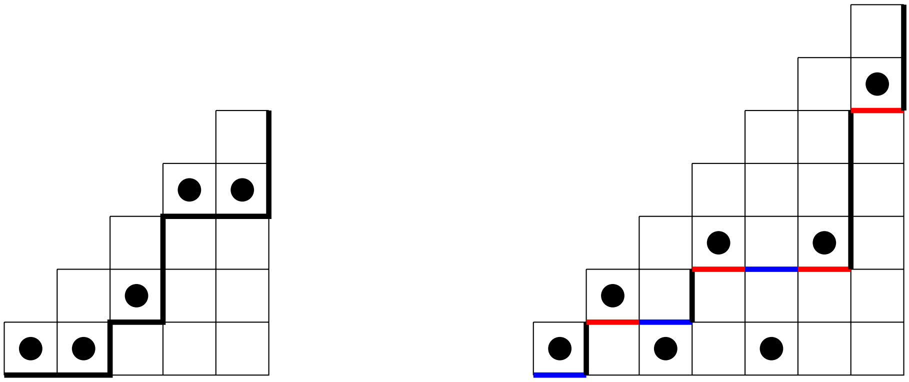

Figure 1: On the left, the 10-avoiding inversion sequence $(0,0,1,3,3)$, and the corresponding Dyck path. On the right, the 021-avoiding inversion sequence $(0,1,0,2,0,2,5)$, and the corresponding coloured Dyck path.

Inversion sequences avoiding 021, and context-free languages

The coloured Dyck paths in bijection with 021-avoiding inversion sequences can be seen as words over the alphabet $\{\mathtt{d},\mathtt{p},\mathtt{z}\}$, where each letter z marks a zero entry of the inversion sequence, p marks a positive entry, and d marks a difference between a left-to-right maximum and the previous one (additionally, a suffix of letters d is placed so that the length of the word is exactly twice that of the inversion sequence). For instance, the inversion sequence $(0,1,0,2,0,2,5)$ of Figure 1 is encoded by the word zdpzdpzpdddpdd. These words form a context-free language, which is recognized by the deterministic counter automaton of Figure 2.

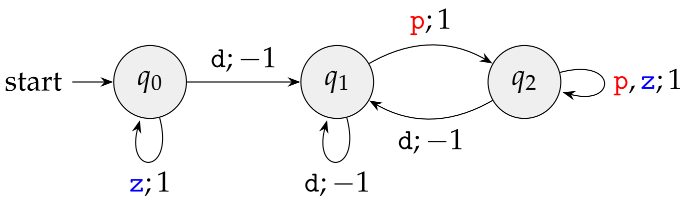

Figure 2: Counter automaton recognizing a language in bijection with 021-avoiding inversion sequences. This automaton accepts a word whenever the counter is 0, and transitions of the form $\ell; -1$ are not allowed when the counter is 0.

The encoding of inversion sequences as coloured Dyck paths is compatible with pattern avoidance in the following sense: a 021-avoiding inversion sequence σ avoids a pattern τ if and only if the Dyck path corresponding to σ avoids one or two patterns associated with τ (they roughly correspond to the encoding of τ as a coloured Dyck path). The previous statement relies on an ad hoc definition of pattern avoidance for words over the alphabet $\{\mathtt{d},\mathtt{p},\mathtt{z}\}$, which satisfies a nice property: the set of words avoiding a pattern is always a rational language.

By classical closure properties of formal languages, it follows that the set of inversion sequences avoiding 021 and any finite set of patterns is in bijection with an unambiguous context-free language. Hence their generating function is algebraic, by the Chomsky–Schützenberger enumeration theorem [1].

The same approach also applies to 021-avoiding ascent sequences, and restricted growth functions. For 021-avoiding restricted growth functions, only rational languages are involved, so the generating functions are even rational.

In addition, with a different encoding of inversion sequences, we show that for any finite set of patterns P which contains 021 and a strictly increasing pattern, the class of P-avoiding inversion (or ascent) sequences has a rational generating function.

References

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