The number of Catalan words avoiding a finite set of patterns has a rational generating function

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There has been much recent research on pattern-avoiding Catalan words. A *Catalan word* is a word $w_1 \dots w_n$ in alphabet $\mathbb{N}$ such that $w_1 = 0$ and $w_{i+1} \leq w_i + 1$ for all i. These are counted by the Catalan numbers, since the function that maps $w_1 \dots w_n$ to the Dyck path with ith up-step at height w_i is a bijection.

A few different kinds of patterns have been studied in the context of Catalan words. Our focus is on what one might call "classical" patterns, as opposed to consecutive or vincular patterns. Under this definition, if $u = u_1 \dots u_k$ and $v = v_1 \dots v_l$ are words in alphabet $\mathbb{N}$, then we say u is *contained* in v if there exist $i_1, \dots, i_k$ such that $\mathrm{st}(u) = \mathrm{st}(v_{i_1} v_{i_2} \dots v_{i_k})$, where $\mathrm{st}(w)$ denotes the standardization of a word w. That is, u is contained in v if v has a (not necessarily consecutive) subsequence with the same relative order as u. This makes the set of Catalan words into a graded poset, just like the graded poset of permutations under pattern containment, whose down-sets are the classes of Catalan words avoiding a given set of Catalan words. Recent research on Catalan words using this definition of pattern containment [4, 3, 5] focuses on systematic enumeration of the Catalan words that avoid one pattern of size 3, two patterns of size 3, or one pattern of size 4, refined by descent number.

We prove that every class of Catalan words avoiding a (non-empty) finite set of patterns has a rational generating function, thus confirming a conjecture of Baril, Mansour, Ramírez, and Shattuck [5]. Our proof uses regular languages and transducers, based on the proof by Albert, Brignall, Ruškuc, and Vatter [2] of the analogous property of 321-avoiding permutations. We also find an infinite antichain of Catalan words, in contrast to the situation of 312-avoiding permutations which have no infinite antichain [1].

References

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