On Patterns in q-Order Statistics

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Vamvakari [7], had introduced q-order statistics (that, is q-analogues of the classical order statistics, for $0 < q < 1$) which arisen from dependent and not identically distributed q-continuous random variables and had studied their distributional properties. She had studied the distribution functions and the q-density functions of the relative q-ordered random variables. In this work, we expand the definition of q-ordered random variables to also include discrete q-distributed random variables. We then show how permutation patterns and permutation statistics arise in the study of the distributional properties of q-ordered random variables.

Brief Introduction

The theory of order statistics and their associated properties has been the subject of extensive investigation over the past several decades. A substantial body of literature is devoted to the analysis of order statistics derived from independent and identically distributed (i.i.d.) random variables. More recently, increasing attention has been directed toward the study of order statistics arising from independent or dependent, but not necessarily identically distributed, random variables—an area that remains of significant theoretical and practical interest. Notably, the study of order statistics shares connections with the theory of permutation patterns, particularly in the context of the relative ordering of sample elements. Understanding the structure and frequency of specific permutations induced by ordered samples contributes to both probabilistic and combinatorial analyses, thereby offering a fruitful interplay between the two fields.

In the field of discrete q-distributions, Charalambides [6, p.167] has presented the order statistics arising from independent and identically distributed random variables, with common distribution a discrete q-uniform distribution. Charalambides [4, 5] also has studied the distributions of the record statistics in q-factorially increasing populations.

Vamvakari [7], had introduced q-order statistics, as q-analogues of the classical order statistics, for $0 < q < 1$, which arisen from dependent and not identically distributed q-continuous random variables and had studied their distributional properties. She had studied the distribution functions and the q-density functions of the relative q-ordered random variables. She had focused focus on q-ordered variables arising from dependent and not identically q-uniformly distributed random variables and she derived their q-distributions, including q-power law, q-beta and q-Dirichlet distributions. The motivation for introducing q-order statistics was given by studying the properties of the waiting times of the Heine process (that, is a q-analogue of the classical Poisson process).

In this work, we expand the definition of q-ordered random variables to also include discrete q-distributed random variables. We then highlight the role of permutation patterns and permutation statistics in the study of distributional properties of the considered q-ordered random variables.

Main Results

Let a ν-variate random variable $\mathcal{X} = (X_1, \ldots, X_\nu)$ be defined in a sample space Ω. Then for the values $x_1 = X_1(\omega), \ldots, x_\nu = X_\nu(\omega)$, $\omega \in \Omega$ there is a permutation $(i_1, \ldots, i_\nu)$ of the ν indices $\{1, \ldots, \nu\}$, such that $x_{i_1} \leq \cdots < x_{i_{\nu-1}} \leq x_{i_\nu}$. The k-th ordered random variable is denoted by $X_{(k)}$ and defined by

$$X_{(k)}(\omega) = x_{(k)},\ \omega \in \Omega,$$

where $x_{(k)} = x_{i_k}$, $k = 1, \ldots, \nu$. In particular, for $k = 1$ this gives $X_{(1)} = \min\{X_1, \cdots, X_\nu\}$ and, for $k = \nu$ this gives $X_{(\nu)} = \max\{X_1, \ldots, X_\nu\}$. Generally, the following inequalities hold:

$$X_{(1)} \leq \cdots \leq X_{(\nu)}.$$

Building on the notion of q-integral introduced by Thomae in 1869, Vamvakari [7], defined q-order statistics-i.e., q-analogues of the classical order statistics- for $0 < q < 1$, which arisen from dependent and not identically distributed q-continuous random variables and had studied their distributional properties. In this work, we expand the definition of q-ordered random variables to also include discrete q-distributed random variables, as follows:

Definition 1 (q-ordered)**.** Let $\mathcal{Y} = (Y_1, \ldots, Y_\nu)$ be a ν-variate random variable and $Y_{(k)}$, $1 \leq k \leq \nu$, be the corresponding k-th ordered random variables. Assume that $Y_{(k)}$, $1 \leq k \leq \nu$, satisfy the inequalities

$$0 \leq Y_{(1)} < qY_{(2)} < Y_{(2)} < \cdots < Y_{(\nu-1)} < qY_{(\nu)} < Y_{(\nu)} \leq \beta, \beta > 0. \quad (1)$$

Then, $Y_{(k)}$ (for any k such that $1 \leq k \leq \nu$) is called the k-th *q-ordered* random variables.

We then show how permutation patterns and permutation statistics arise in the study of the distributional properties of q-ordered random variables. Analytically, we present generalized versions of theorems originally established by the author in 2023, highlighting the role of permutation patterns and permutations statistics in our study.

Theorem 2. *Let $Y_1, \ldots, Y_\nu$ be dependent random variables, where $Y_(i), i = 1, 2, \ldots, \nu$, satisfy inequalities (1). Then, the distribution function of the k-th q-ordered random variable $Y_{(k)}$, $1 \leq k \leq \nu$, is given for $y \in [0, \beta]$ by*

$$\begin{aligned} F_{Y_{(k)}}(y) &= P\left(Y_{(k)} \leq y\right) \\ &= \sum_{r=k}^{\nu} \sum_{1 \leq i_1 < \ldots < i_r \leq \nu} \prod_{j=1}^{r} F_{Y_{i_j}}\left(q^{j-1}y\right) \prod_{m=r+1}^{\nu} \left(1 - F_{Y_{i_m}}\left(q^{i_m-(m-r)}y\right)\right), \quad (2) \end{aligned}$$

where the inner summation is over all r-combinations $\{i_1, \ldots, i_r\}$ of the set $\{1, \ldots, \nu\}$ and $F_{Y_i}(y) = P\left(Y_i \leq y\right)$ is the distribution function of each Y_i, $i = 1, 2, \ldots, \nu$.

Theorem 3. *Let $Y_1, \ldots, Y_\nu$ be dependent random variables, where $Y_(i), i = 1, 2, \ldots, \nu$, satisfy inequalities (1). Then, the joint distribution function of the q-ordered random* *variables $Y_{(k)}$ and $Y_{(r)}$ for $1 \leq k < r \leq \nu$, is given for $y, z \in [0, \beta]$ by*

$$F_{Y_{(k)},Y_{(r)}}(y,z) = P\left(Y_{(k)} \leq y, Y_{(r)} \leq z\right)$$
$$= \sum_{j=r}^{\nu} \sum_{s=k}^{j} \sum \prod_{n_1=1}^{s} F_{Y_{i_{n_1}}}\left(q^{n_1-1} y\right) \prod_{n_2=s+1}^{j} \left(F_{Y_{i_{n_2}}}\left(q^{n_2-s-1} z\right) - F_{Y_{i_{n_2}}}(y)\right)$$
$$\cdot \prod_{n_3=j+1}^{\nu} \left(1 - F_{Y_{i_{n_3}}}\left(q^{i_{n_3}-(n_3-j)} z\right)\right), y < q^{r-k} z, 1 \leq k < r \leq \nu, \quad (3)$$

where the inner summation is over all pairwise disjoint subsets $\{i_1, \ldots, i_s\}$ and $\{i_{s+1}, \ldots, i_j\}$ of the set $\{1, \ldots, \nu\}$ with $1 \leq i_1 < \cdots < i_s \leq \nu$ and $1 \leq i_{s+1} < i_{s+2} < \cdots < i_j \leq \nu$, and $F_{Y_i}(y) = P(Y_i \leq y)$ is the distribution function of each Y_i, $i = 1, 2, \ldots, \nu$.

Remark 4. The above theorems yield, among other results, distributional properties of order statistics arising from dependent and non-identically distributed random variables, specifically those following q-uniform or discrete q-uniform distributions.

References

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