Enumerating 1324-avoiders with few inversions

Emil Verkama KTH Royal Institute of Technology

This talk is based on joint work with Svante Linusson

In 2012, Claesson, Jelínek and Steingrímsson tackled the notoriously difficult problem of enumerating the 1324-avoiders, leading to the first interesting upper bound for their Stanley–Wilf limit [2]. They also formulated the *inversion-monotonicity conjecture*, whose proof would improve the upper bound even further. In this work, we prove half of the conjecture. Let $\mathrm{Av}_n^k(\tau)$ denote the set of τ-avoiders of length n with k inversions, and $\mathrm{av}_n^k(\tau) = |\mathrm{Av}_n^k(\tau)|$.

Conjecture 1 (Conjecture 13 in [2])**.** *For all nonnegative integers n and k,*

$$\mathrm{av}_n^k(1324) \leq \mathrm{av}_{n+1}^k(1324).$$

Any $\pi \in \mathfrak{S}_n$ with $n \geq \mathrm{inv}(\pi) + 2$ is (sum) decomposable. A decomposable 1324-avoider is necessarily of the form

$$\pi = \pi^{(1)} \oplus 1 \oplus 1 \oplus \ldots \oplus 1 \oplus \pi^{(2)},$$

where $\pi^{(1)} \in \mathrm{Av}(132)$ and $\pi^{(2)} \in \mathrm{Av}(213)$; therefore

$$\mathrm{av}_n^k(1324) = \sum_{i=0}^{k} p(k)p(k-i) =: a(k) \qquad \text{for all } n \geq k+2, \tag{1}$$

where $p(k)$ is the number of integer partitions of k. Using the classical estimate $p(k) < \exp\left(\pi\sqrt{2k/3}\right)$ along with (1), one can show that Conjecture 1 implies the bound

$$L(1324) < \exp\left(\pi\sqrt{2/3}\right) \approx 13.001594.$$

This would still today be the best known upper bound for the Stanley–Wilf limit of the 1324-avoiders: the current records are $10.27 \leq L(1324) \leq 13.5$ due to Bevan, Brignall, Elvey Price and Pantone [1]. Convincing estimates by Conway, Guttmann and Zinn-Justin point at the actual value being $L(1324) \approx 1.600 \pm 0.003$ [3].

Since 2012, Conjecture 1 has received much attention. Our main result marks the first bit of progress: the conjecture holds for all k and n such that $n \geq \frac{k+7}{2}$. The proof relies on a structural characterization of the permutations in question in terms of a new notion of *almost-decomposability*, leading to an enumeration.

Theorem 2. *For all nonnegative integers k and $n \geq \frac{k+7}{2}$,*

$$\mathrm{av}_n^k(1324) = a(k) - 4a(k-n+1) - 6\sum_{i=0}^{k-n} a(i), \tag{2}$$

where $a(k) = \sum_{i=0}^{k} p(i)p(k-i)$ and $p(k)$ is the number of integer partitions of k. In particular,

$$\mathrm{av}_{n+1}^k(1324) - \mathrm{av}_n^k(1324) = 4a(k-n+1) + 2a(k-n) \geq 0,$$

and this difference has a combinatorial interpretation.

Remark 3. In the language of generating functions, Theorem 2 states that

$$\mathrm{av}_n^k(1324) = [x^k]\left(P(x)^2 - \frac{R_n(x)}{1-x}\right)$$

whenever $n \geq \frac{k+7}{2}$, where $P(x) = \sum_{i\geq 0} p(i)x^i$ is the generating function for the partition numbers and $R_n(x) = 2(2+x)x^{n-1}P(x)^2$. In particular,

$$\mathrm{av}_{n+1}^k(1324) - \mathrm{av}_n^k(1324) = [x^k]R_n(x).$$

This wording is more convenient for the proof.

Observe that Conjecture 1 holds (with equality) for all k and n such that $n \geq k+2$ due to (1). It therefore remained to prove that the conjecture holds when $n \leq k+1$, and after Theorem 2, only the part $n < \frac{k+7}{2}$ remains. Table 1 shows the values of $\mathrm{av}_n^k(1324)$ for $n \leq 20$, $k \leq 25$. The constant part of each column is colored in blue; the sequence $1, 2, 5, 10, 20, \ldots$ is given by $a(k)$ from (1). Theorem 2 gives the values contained in the red cells with formula (2). The uncolored cells mark the region for which Conjecture 1 is still open. A larger version of the table is available at `https://akc.is/inv-mono/` courtesy of Anders Claesson.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	1																									
2	1	1																								
3	1	2	2	1																						
4	1	2	5	6	5	3	1																			
5	1	2	5	10	16	20	20	15	9	4	1															
6	1	2	5	10	20	32	51	67	79	80	68	49	29	14	5	1										
7	1	2	5	10	20	36	61	96	148	208	268	321	351	347	308	241	165	98	49	20	6	1				
8	1	2	5	10	20	36	65	106	171	262	397	568	784	1019	1264	1478	1628	1681	1619	1441	1173	866	574	338	174	76
9	1	2	5	10	20	36	65	110	181	286	443	664	985	1416	1988	2715	3589	4579	5631	6654	7559	8225	8545	8457	7930	7006
10	1	2	5	10	20	36	65	110	185	296	467	714	1077	1582	2305	3284	4617	6374	8665	11521	15012	19067	23599	28426	33300	37862
11	1	2	5	10	20	36	65	110	185	300	477	738	1127	1682	2477	3584	5134	7240	10100	13915	18976	25563	34017	44640	57739	73421
12	1	2	5	10	20	36	65	110	185	300	481	748	1151	1732	2577	3768	5450	7766	10976	15312	21171	28973	39338	52919	70657	93482
13	1	2	5	10	20	36	65	110	185	300	481	752	1161	1756	2627	3868	5634	8098	11526	16216	22632	31266	42845	58213	78531	105137
14	1	2	5	10	20	36	65	110	185	300	481	752	1165	1766	2651	3918	5734	8282	11858	16786	23568	32768	45234	61902	84130	113477
15	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2661	3942	5784	8382	12042	17118	24138	33728	46776	64346	87939	119306
16	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2665	3952	5808	8432	12142	17302	24470	34298	47736	65916	90431	123184
17	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2665	3956	5818	8456	12192	17402	24654	34630	48306	66876	92001	125708
18	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2665	3956	5822	8466	12216	17452	24754	34814	48638	67446	92961	127278
19	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2665	3956	5822	8470	12226	17476	24804	34914	48822	67778	93531	128238
20	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2665	3956	5822	8470	12230	17486	24828	34964	48922	67962	93863	128808

Table 1: The numbers $\mathrm{av}_n^k(1324)$.

Our proof of Theorem 2 uses a structural characterization of 1324-avoiders with few inversions. We say that a permutation $\pi \in \mathfrak{S}_n$ is *almost decomposable* if it is indecomposable, but

$$\max\{\mathrm{comp}(\pi \setminus e) : e \in \{1, \pi_1, n, \pi_n\}\} \geq 2. \tag{3}$$

Here $\pi \setminus \pi_i$ denotes the reduction of $\pi_1 \ldots \pi_{i-1}\pi_{i+1} \ldots \pi_n$, and $\mathrm{comp}(\pi)$ is the number of sum components of π.

Theorem 4. *Every indecomposable* 1324*-avoider* π *such that* $\mathrm{inv}(\pi) \leq 2|\pi| - 7$ *is almost decomposable.*

Almost-decomposability allows us to construct an injection $\mathrm{Av}^k_{n+1}(1324) \to \mathrm{Av}^k_n(1324)$ in a natural way when $n \geq \frac{k+7}{2}$. If $\pi \in \mathrm{Av}(1324)$ is decomposable, it is of the form

$$\pi = \pi^{(1)} \oplus \underbrace{1 \oplus 1 \oplus \ldots \oplus 1}_{m \text{ times}} \oplus \pi^{(2)},$$

and we can set

$$f(\pi) = \pi^{(1)} \oplus \underbrace{1 \oplus 1 \oplus \ldots \oplus 1}_{m+1 \text{ times}} \oplus \pi^{(2)}.$$

On the other hand, if π is indecomposable but $\pi \setminus \pi_1$ is decomposable, we let $f(\pi)$ be the permutation σ such that $\sigma_1 = \pi_1$ and $\sigma \setminus \sigma_1 = f(\pi \setminus \pi_1)$. This is easily extended to the other cases of (3) with symmetries, and f preserves both 1324-avoidance and inversion number.

Theorem 5. *Let k and n be nonnegative integers such that $n \geq \frac{k+7}{2}$. The map*

$$f : \mathrm{Av}^k_n(1324) \longrightarrow \mathrm{Av}^k_{n+1}(1324)$$

we defined above is injective.

The remainder of the proof of Theorem 2 relies on a careful enumeration of the collection of permutations $\mathrm{Av}^k_{n+1}(1324) \setminus \mathrm{im}\, f$. Complete proofs of Theorems 2, 4 and 5 are available in our preprint [4]. At least two natural questions arise:

- Can our method be extended to prove a larger part of Conjecture 1?
- Does almost-decomposability have other uses?

References

[1] D. Bevan, R. Brignall, A. Elvey Price, and J. Pantone. “A structural characterisation of Av(1324) and new bounds on its growth rate”. In: *European J. Combin.* 88 (2020), 103115.

[2] A. Claesson, V. Jelínek, and E. Steingrímsson. “Upper bounds for the Stanley-Wilf limit of 1324 and other layered patterns”. In: *J. Combin. Theory, Ser. A* 119.8 (2012), pp. 1680–1691.,

[3] A. R. Conway, A. J. Guttmann, and P. Zinn-Justin. “1324-avoiding permutations revisited”. In: *Adv. in Appl. Math.* 96 (2018), pp. 312–333.

[4] S. Linusson and E. Verkama. “Enumerating 1324-avoiders with few inversions”. arXiv:2408.15075 (2024).