Exhaustive Generation of Pattern-Avoiding s-Words

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This talk is based on joint work with Samuel Buick, Madeleine Goertz, Amos Lastmann, Kunal Pal, Helen Qian, Sam Tacheny, Leah Williams, and Yulin Zhai (NSF Grant DMS2241623)

We introduce a simple approach to generating Gray codes of pattern-avoiding s-words (i.e., multiset permutations) and corresponding combinatorial objects. It generalizes plain changes and the recent *Combinatorial Generation via Permutation Language* series.

Introduction *Plain changes* is a swap Gray code for S_n the permutations of [n] = {1,...,n}: e.g., 123, 132, 312, 321, 231, 213 where a *swap* moves a large digit past a small digit. It was discovered in the 1600s and is also known as the *Steinhaus-Johnson-Trotter algorithm*. More recently, it has been viewed as a greedy algorithm: "swap the largest value" [5].

A *permutation language* L is a subset of S_n. A *jump* moves a larger digit past **one or more** smaller digits. It is *minimal* for $w \in L$ if its *distance* d (i.e., the number of smaller digits) is minimized to create $w' \in L$. Algorithm J was introduced at *Permutation Patterns 2019*: "minimal jump the largest value" [2]. It generates zig-zag languages e.g., 123, 132, 312, 321, 213 for $\mathsf{Av}_3(231)$. But jumps are limited when working with *s-words* (or *s-permutations*) which have s_i copies of i for $i \in [m]$. For example, $w = 123331$ is a *Stirling s-word* for $s = (2,1,3)$ (i.e., $w \in \mathsf{Av}_s(212)$) and every jump applied to w is *invalid* (i.e., the jump creates a 212 pattern) so the associated flip graph is disconnected. Each $v \in [m]$ is a *value* and each copy of a value in w is a *digit*. The set of all s-words is S_s.

We consider "bumps" which move a **run** of a larger digit. Algorithm B generates many *s-word languages* (i.e., subsets of S_s). Some applications are below and in Figures 1–2.

(a) Gray codes for S_s (e.g., 1122,1221,1212,2112,2121,2211 for $s = (2,2)$) using transpositions.
(b) *Stirling changes* generalizes plain changes to *Stirling s-words* $\mathsf{Av}_s(212)$ using transpositions.
(c) Bump Gray code for regular words counted by k-Catalan numbers $\mathsf{Av}_m^{k-1}(132,121)$ [3, 6].

Our Gray codes lead to efficient algorithms. For example, we generate (b) *looplessly* (i.e., worst-case $O(1)$-time per word) in Algorithm 1. Like the *Permutation Languages* series they also have many applications. For example, (b) leads to a Gray code for s-increasing trees that proves Theorem 1, while (c) gives Gray codes for various k-Catalan objects.

Theorem 1. *Every s-permutohedron has a Hamilton path. (See [1] but with inverted values.)*

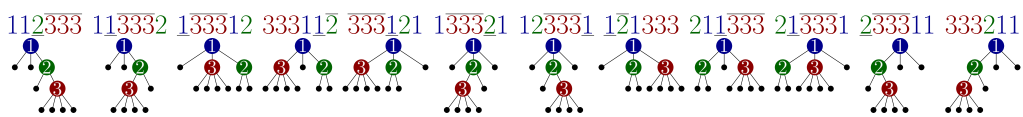

Figure 1: Stirling changes for Stirling s-words with $s = (2,1,3)$ (i.e., permutations of {1,1,2,3,3,3} avoiding 212) as generated by Algorithm B with its corresponding s-increasing tree Gray code.

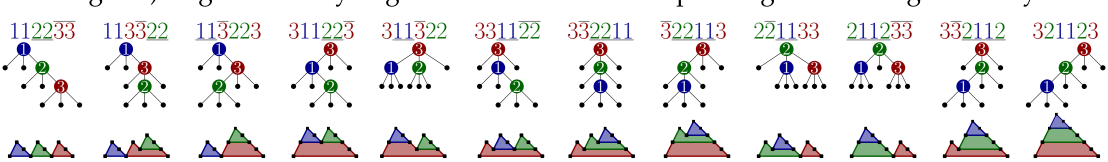

Figure 2: Pattern-avoiding s-words $\mathsf{Av}_s(132,121)$ for $s = (2,2,2)$ with Gray codes for 3-Catalan objects. The ternary trees differ by edge moves preserving inorder (visit self before last child).

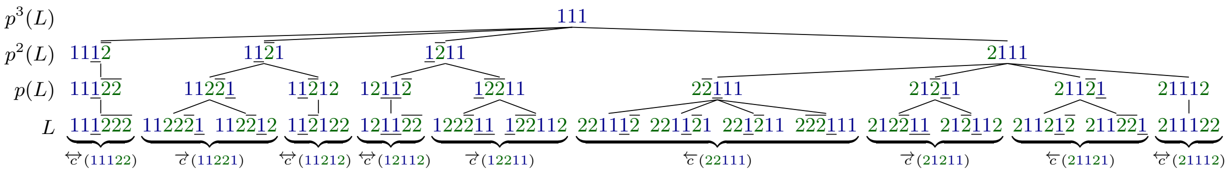

Figure 3: Algorithm B's Gray code for $L = \mathsf{Av}_{3,3}(12121)$ and its ancestor languages. Successive children jump the rightmost largest value while a bump changes the last child of one parent to the first child of the next parent. Note that the rightmost largest value may join these bumps. For example, the jump $2\overline{2}\underline{1}11 = 21211$ in $p(L)$ widens to the bump $2\overline{22}\underline{1}11 = 212211$ in L.

Algorithm B for Bumps Let $w = w_1 w_2 \cdots w_n$ be an s-word with $s = (s_1, s_2, \ldots, s_m)$. The *right-run at index i* is $w_i w_{i+1} \cdots w_j$ with $w_i = \cdots = w_j$ and $j = n$ or $w_j \neq w_{j+1}$. (i.e., a right-maximal run). A *right-bump* moves a right-run to the right past some smaller digits. *Left-run* and *left-bump* are similar. A bump's *value v*, *width w*, *distance d*, and *index i* are its larger digit, # of larger digits, # of smaller digits, and run index. For example, $11\underline{211}\overline{33}322331$ creates $1\overline{33}\underline{1211}333221$ by a left-bump of $v = 3$, $w = 2$, $d = 4$, $i = 7$. Bumps can be jumps ($w = 1$), transpositions ($d = 1$), or swaps ($w = d = 1$).

Let L be an s-word language. Suppose a bump at index i changes $w \in L$ to w'. The bump is *minimal* for its index and direction if it has the shortest distance with $w' \in L$. Minimal bumps on $w \in L$ are uniquely determined by their index and direction. Our generalization of Algorithm J [2] carefully prioritizes values, indices, and directions.

Algorithm *B (Greedy Bumps)*. This algorithm attempts to generate an s-word language L with $s = (s_1, s_2, \ldots, s_m)$ starting from a given or default initial word $w \in L$.
B1. [Initialize] Visit the given initial word w (or by default visit $w = 1^{s_1} 2^{s_2} \cdots m^{s_m}$).
B2. [Greedy] Let w be the most recent word visited. Visit a new word by applying to w a minimal bump prioritized by largest value, then largest index, then rightward over leftward. Halt if no such bump exists. Otherwise, repeat B2.

A bump is *maximum* if it uses the longest distance from an index in a direction. A *zig-zag language* is an s-word language closed under maximum bumps (c.f., [2]). This includes $\mathsf{Av}_s(\alpha)$ when α is *tame*: its largest values are *internal* (i.e., not first or last) and *isolated* (i.e., not consecutive). Zig-zag languages are closed under intersection (and union) so the *peakless s-words* $\mathsf{Av}_s(132,231,121)$ are a zig-zag language.

Theorem 2.[1] *If L is zig-zag language of s-words for $s = (s_1, s_2, \ldots, s_m)$, then Algorithm B generates a bump Gray code for L starting from the non-decreasing word $w = 1^{s_1} 2^{s_2} \cdots m^{s_m} \in L$.*
Sketch. Consider the end of an inductive argument on $n = \sum s$. The *parent* $p(s)$ of s is $(s_1, s_2, \ldots, s_m - 1)$ if $s_m > 1$ or $(s_1, s_2, \ldots, s_{m-1})$ if $s_m = 1$. Similarly, $p(w)$ removes the rightmost m from $w \in L$, and $p(L) = \{p(w) \mid w \in L\}$. The *children* of $w' \in p(L)$ are $c(w') = \{w \in L \mid p(w) = w'\}$. Since L is a zig-zag language, $c(w')$ has s-words where the rightmost m is at (a) index n, and (b) index 1 if $s_m = 1$ or beside the rightmost m in w' if $s_m > 1$; these extremes are equal when w' ends in m. Let $\overrightarrow{c}(w')$ list $c(w')$ in lexicographic order (i.e., the rightmost m jumps left-to-right) and $\overleftarrow{c}(w')$ in reverse; use $\overleftrightarrow{c}(w')$ when w' has one child. Note that $p(L)$ is a zig-zag language so at some point Algorithm B generates w'. We claim that Algorithm B generates $\overrightarrow{c}(w')$ or $\overleftarrow{c}(w')$ (or $\overleftrightarrow{c}(w')$) when run on L. It also applies a bump from the last child of one parent to the first child of the next parent. In other words, it uses *local recursion*. See Figure 3. □

[1] This is not a full characterization as Algorithm B generates some non-zig-zag language e.g., $\mathsf{Av}_s(212)$.

Algorithm 1 Loopless generation of Stirling s-words $\mathrm{Av}_s(212)$ in Stirling changes order.

```
def Stirling(s1, s2, ..., sm)            ▷ 1-based indexing
  t1, t2, t3, ..., tm ← 0, s1, s1+s2, ..., s1+···+s(m−1)
  perm ← 1^s1 2^s2 ··· m^sm               ▷ s-word (s-perm)
  left ← [t1+1, ..., t(m−1)+1]            ▷ left indices 1,...,m
  inv ← 0^m   ▷ # of smaller digits right of 1,2,...,m
  fs ← 12···m ▷ focus pointers for 1,...,m (see [4])
  dirs ← −1^m                             ▷ bump directions for 1,...,m
  v ← fs[m]                               ▷ larger value in next bump
  while v > 1 do                          ▷ 1 is never the larger value
    d ← dirs[v]                           ▷ direction of v
    if d = 1 then                         ▷ right-bump v's run?
      i ← left[v]                         ▷ first index of v
      j ← left[v] + s[v]                  ▷ index after v's run
    else                                  ▷ left-bump v's run
      i ← left[v] + s[v] − 1              ▷ last index of v
      j ← left[v] − 1                     ▷ index before v's run
    u ← perm[j]                           ▷ smaller digit to bump
    visit perm                            ▷ next s-word in Gray code
    perm[i] ← u                           ▷ apply bump to the s-word
    perm[j] ← v                           ▷ (as a transposition)
    left[v] ← left[v]+d                   ▷ leftmost v moved
    if left[u] = j then                   ▷ leftmost u moved?
      left[u] ← left[u]−d·s_v             ▷ u passed all v
    inv[v] ← inv[v]−d                     ▷ v's run passed one u
    if inv[v] = 0 or inv[v] = t_v then    ▷ v limit?
      dirs[v] ← −d                        ▷ change v's direction
      fs[v] ← fs[v−1]                     ▷ inherit focus of v−1
      fs[v−1] ← v−1                       ▷ reset focus of v−1
    v ← fs[m]                             ▷ larger value in next bump
    fs[m] ← m                             ▷ reset focus of m
  visit perm                              ▷ last s-word in Gray code
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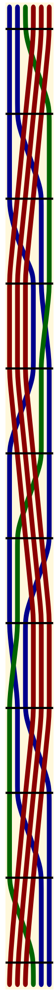
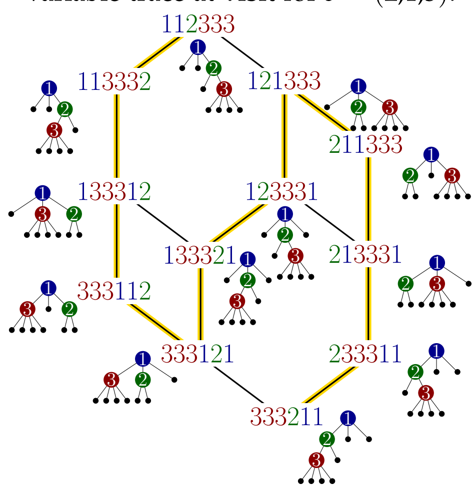


perm	v	u	i	j	left	inv	fs	dirs
112333	3	2	6	3	134	000	123	---
113332	3	1	5	2	163	001	123	---
133312	3	1	4	1	162	002	123	---
333112	2	1	6	5	461	003	123	--+
333121	3	1	1	4	451	013	123	--+
133321	3	2	2	5	152	012	123	--+
123331	3	1	3	6	123	011	123	--+
121333	2	1	2	1	124	010	123	---
211333	3	1	6	3	214	020	113	-+-
213331	3	1	5	2	213	021	113	-+-
233311	3	2	4	1	512	022	113	-+-
333211	1				541	023	123	-++

Variable trace at **visit** for $s = (2,1,3)$.



Hamilton path (2,1,3)-permutohedron (or $s = (3,1,2)$ for decreasing trees [1]).

References

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