Permutations with only reduced co-Bumpless Pipe Dreams

Joshua Arroyo
joint work with Adam Gregory
University of Florida

July 7, 2025

Bumpless Pipe Dreams

Definition (Bumpless Pipe Dream)

A *bumpless pipe dream (BPD)* is a $n \times n$ filling using the tiles

 and 

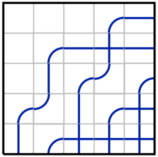
so that pipes enter from the bottom and exit out of the right. Each bumpless pipe dream has an associated permutation given by the order of its pipes.

Lam–Lee–Shimozono (2018) // Weigandt (2021)

Alternating Sign Matrices

Bumpless pipe dreams are in bijection with alternating sign matrices taking:

-  $\mapsto$ 1
-  $\mapsto$ -1
- else $\mapsto$ 0

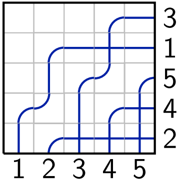


$$\mapsto \begin{array}{|ccccc|}\hline 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline\end{array}$$

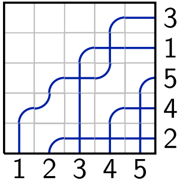
Nonreduced Bumpless Pipe Dreams

Definition (Reduced)

A BPD is *reduced* when each pair of pipes cross at most once.



Reduced



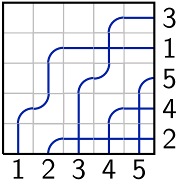
Nonreduced

For the permutation of a nonreduced BPD we treat crossings past the first as the pipes “bumping” off each other instead of crossing.

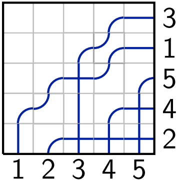
Nonreduced Bumpless Pipe Dreams

Definition (Reduced)

A BPD is *reduced* when each pair of pipes cross at most once.



Reduced

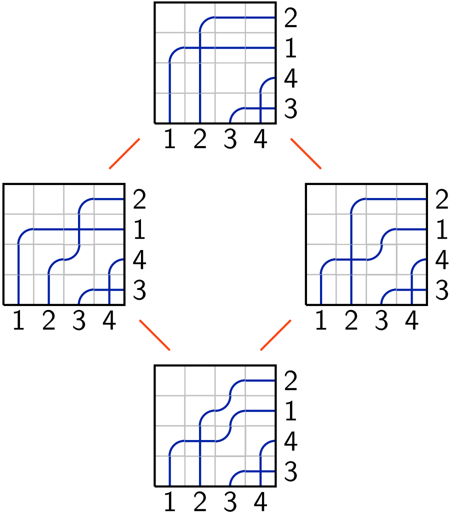


Nonreduced

For the permutation of a nonreduced BPD we treat crossings past the first as the pipes “bumping” off each other instead of crossing.

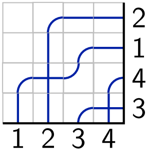
Droop Moves

All bumpless pipe dreams of a given permutation are connected by droop moves.



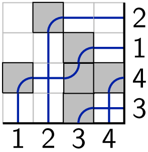
co-BPD

For B in BPD(w), corresponding co-BPD is



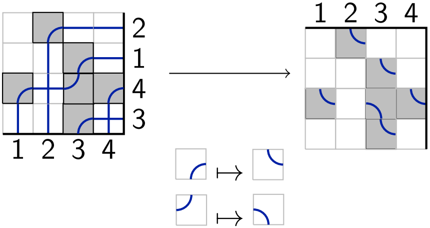
co-BPD

For B in BPD(w), corresponding co-BPD is



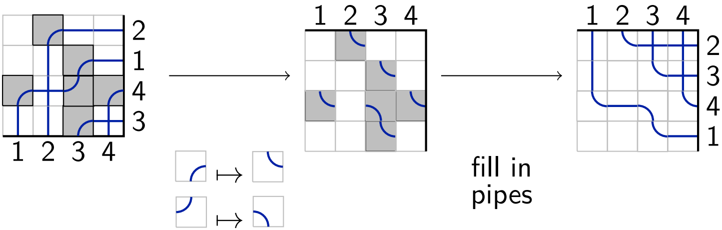
co-BPD

For B in $\mathrm{BPD}(w)$, corresponding co-BPD is



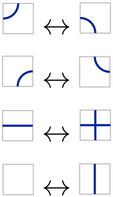
co-BPD

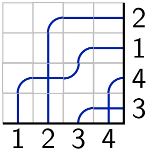
For B in $\mathrm{BPD}(w)$, corresponding co-BPD is

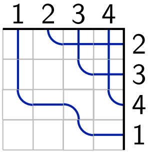


co-BPD

Equivalently $B \leftrightarrow \text{co}(B)$ via the mapping



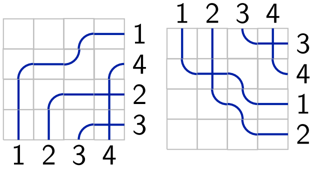




Question

Open problem (Weigandt 2024 at IPAM)

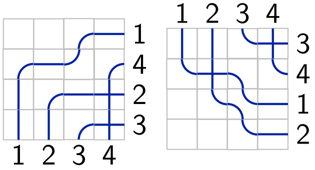
Characterize w for which co(B) is *reduced* for every $B \in \mathrm{BPD}(w)$.



Question

Open problem (Weigandt 2024 at IPAM)

Characterize w for which co(B) is *reduced* for every $B \in \mathrm{BPD}(w)$.



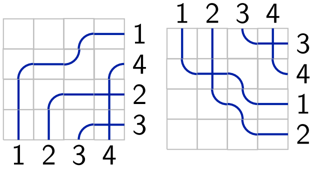
Answer

w has this property if and only if w avoids the seven patterns 1423, 12543, 13254, 25143, 215643, 216543, and 241653.

Question

Open problem (Weigandt 2024 at IPAM)

Characterize w for which co(B) is *reduced* for every $B \in \mathrm{BPD}(w)$.



Answer

w has this property if and only if w avoids the seven patterns 1423, 12543, 13254, 25143, 215643, 216543, and 241653.

Motivation comes from reduced co-BPDs being used to compute a change of basis from Grothendieck to Schubert polynomials.

Schubert Polynomials

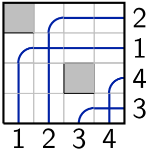
Definition (Schubert Polynomial)

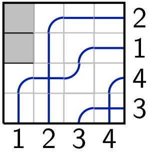
A Schubert polynomial of a permutation w is

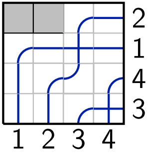
$$\mathfrak{S}_w = \sum_{B \in \mathrm{bpd}(w)} wt(B),$$

where $\mathrm{bpd}(w)$ is the set of all reduced BPDs of w and each empty box in row i has weight x_i.

Schubert Polynomials







$$\mathfrak{S}_{2143} = x_1x_3 + x_1x_2 + x_1^2$$

Grothendieck Polynomials

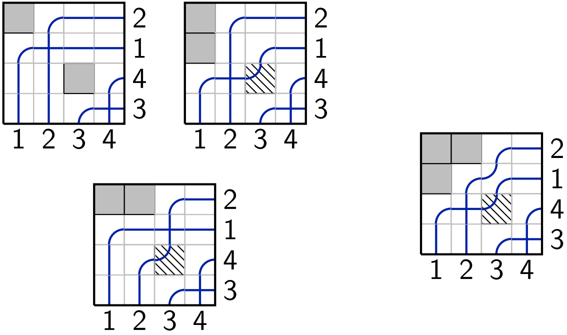
Definition (Grothendieck Polynomial)

A Grothendieck polynomial of a permutation w is

$$\mathfrak{G}_w = \sum_{B \in \mathrm{BPD}(w)} (-1)^{\ell(w)} wt(B),$$

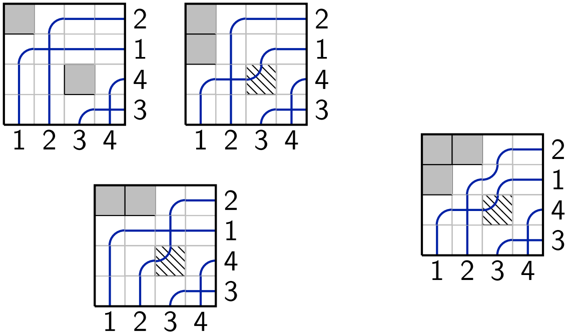
where $\mathrm{BPD}(w)$ is the set of *all* BPDs of w and each empty box in row i has weight $-x_i$ and each  in row i has weight $1 - x_i$.

Grothendieck Polynomials



$$\mathfrak{G}_{2143} = x_1x_3 + (x_1x_2 - x_1x_2x_3) + (x_1^2 - x_1^2x_3) + (-x_1^2x_2 + x_1^2x_2x_3)$$

Schubert and Grothendieck Polynomials



$$\mathfrak{G}_{2143} = x_1x_3 + (x_1x_2 - x_1x_2x_3) + (x_1^2 - x_1^2x_3) + (-x_1^2x_2 + x_1^2x_2x_3)$$
$$= \mathfrak{S}_{2143} + \text{higher order terms}$$

Grothendieck into Schubert

Theorem (Weigandt, 2025)

Let $\text{co-BPD}(w) = \{\text{co}(B) : B \in \text{BPD}(w), \text{co}(B) \textit{ is reduced}\}$ *and for a* co-BPD, C, *let* $\pi(C)$ *be the permutation associated with* C. *Then,*

$$\mathfrak{G}_w = \sum_{C \in \text{co-BPD}(w)} (-1)^{\ell(\pi(C)) - \ell(w)} \cdot \mathfrak{S}_{\pi(C)}.$$

Grothendieck into Schubert

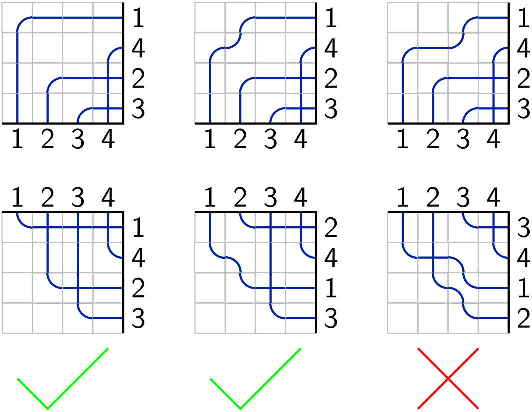
Theorem (Weigandt, 2025)

Let $\text{co-BPD}(w) = \{\text{co}(B) : B \in \text{BPD}(w), \text{co}(B) \textit{ is reduced}\}$ *and for a* co-BPD, C, *let* $\pi(C)$ *be the permutation associated with* C*. Then,*

$$\mathfrak{G}_w = \sum_{C \in \text{co-BPD}(w)} (-1)^{\ell(\pi(C)) - \ell(w)} \cdot \mathfrak{S}_{\pi(C)}.$$

This is a BPD variant of a theorem by Lenart (1999).

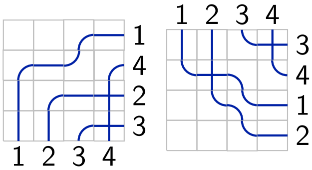
$$\mathfrak{G}_{1423} = \mathfrak{S}_{1423} - \mathfrak{S}_{2413}$$



Question

Open problem (Weigandt 2024 at IPAM)

Characterize w for which co(B) is *reduced* for every $B \in \mathrm{BPD}(w)$.

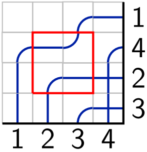


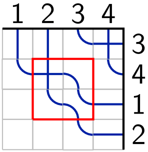
Answer

w has this property if and only if w avoids the seven patterns 1423, 12543, 13254, 25143, 215643, 216543, and 241653.

Observation

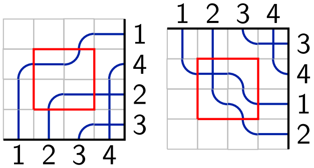
What does a double-crossing in a co-BPD look like?



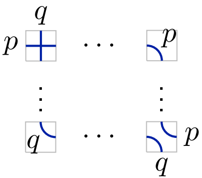


Observation

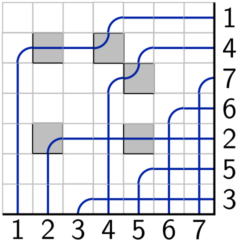
What does a double-crossing in a co-BPD look like?



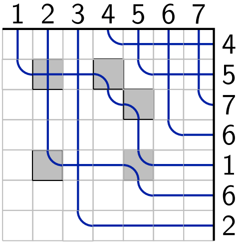
More generally,



Example



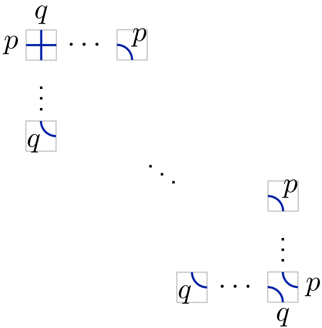
B

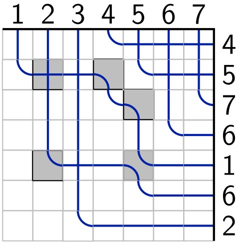


co(B)

Observation

Actually, a co-BPD has a double-crossing if and only if it has:

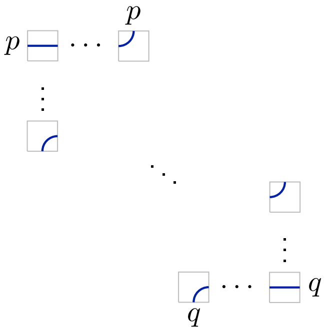


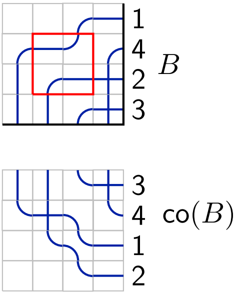


co(B)

Observation

If a co-BPD is non-reduced, its corresponding BPD has:





Call this "*the configuration*".

Outline

Property: "co(B) reduced for every $B \in$ BPD(w)"

w has property $\iff$ none of its BPDs have the config.

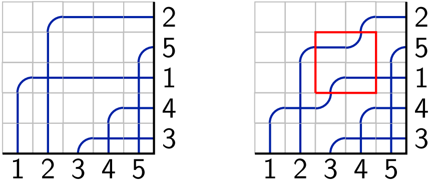
$\iff w$ avoids those seven patterns.

(Nec.) if w contains a pattern then it has a BPD with config.

(Suff.) if $B \in$ BPD(w) has config then w contains a pattern.

Necessary to Avoid

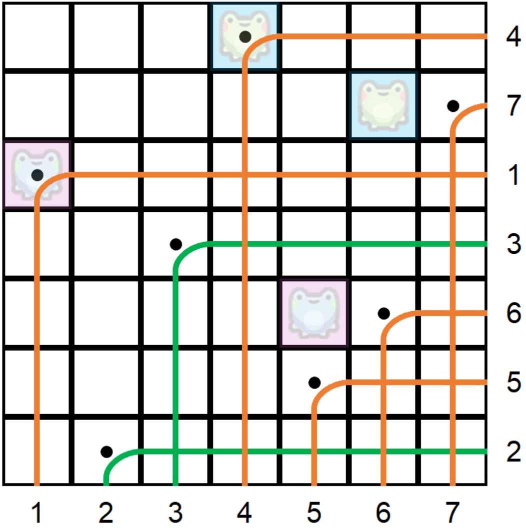
Rothe BPD is unique BPD with no  tiles.



Perform droop moves from Rothe to get a BPD with config.

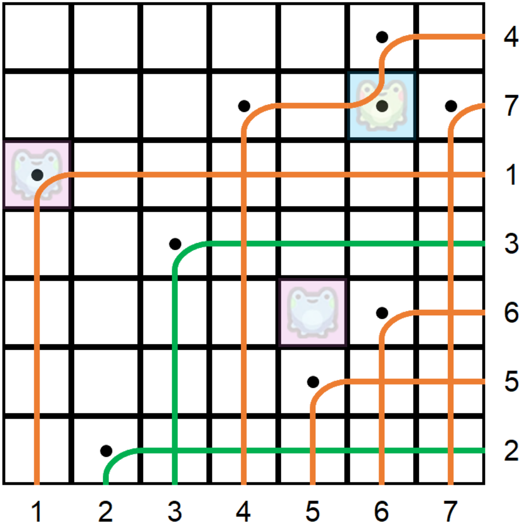
Necessary to Avoid

Perform droop moves from Rothe to get a BPD with config.



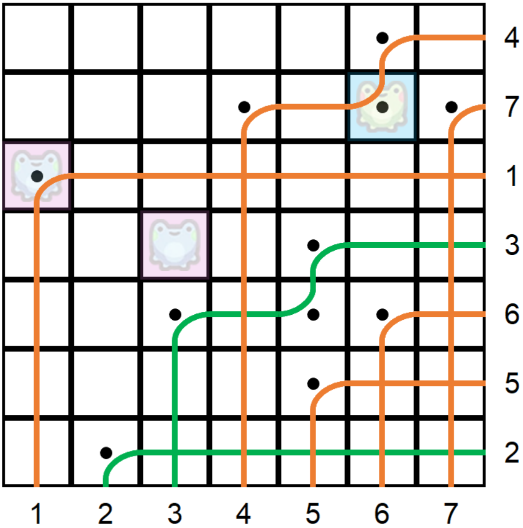
Necessary to Avoid

Perform droop moves from Rothe to get a BPD with config.



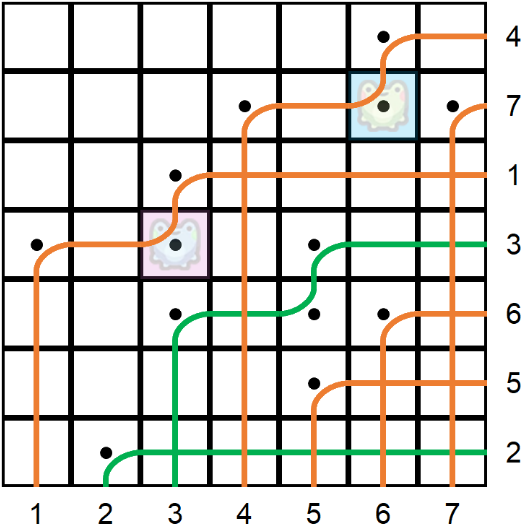
Necessary to Avoid

Perform droop moves from Rothe to get a BPD with config.



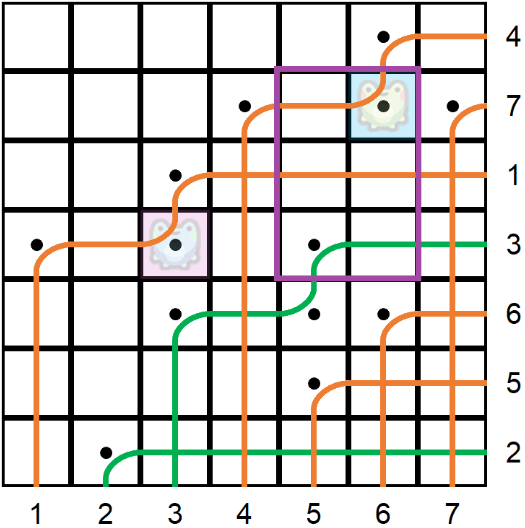
Necessary to Avoid

Perform droop moves from Rothe to get a BPD with config.



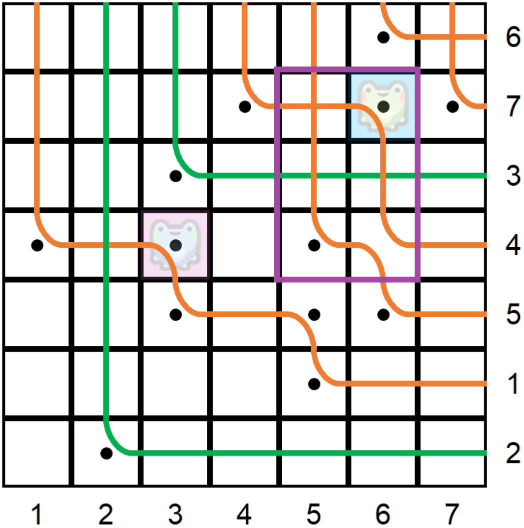
Necessary to Avoid

Perform droop moves from Rothe to get a BPD with config.



Necessary to Avoid

Perform droop moves from Rothe to get a BPD with config.



Outline

Property: "co(B) reduced for every $B \in$ BPD(w)"

w has property $\iff$ none of its BPDs have the config.

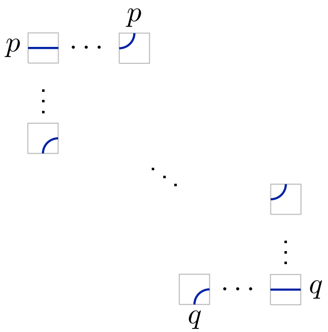
$\iff$ w avoids those seven patterns.

(Nec.) if w contains a pattern then it has a BPD with config.

(Suff.) if $B \in$ BPD(w) has config then w contains a pattern.

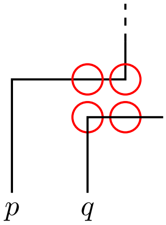
Sufficient to Avoid

Suppose $B \in \mathrm{BPD}(w)$ has the configuration.

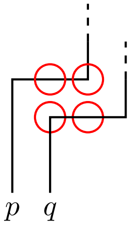


Sufficient to Avoid

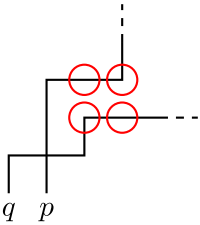
We consider three cases:



don't cross before &
q doesn't droop after



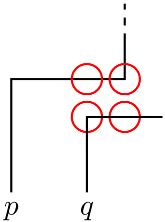
don't cross before &
q does droop after



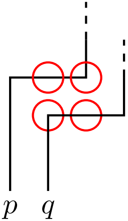
cross before

Sufficient to Avoid

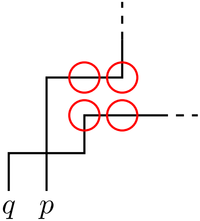
We consider three cases:



1423
13254



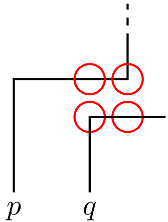
1423, 12543
215643, 216543



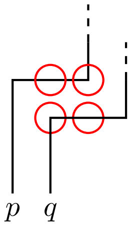
1423, 12543, 25143
216543, 241653

Sufficient to Avoid

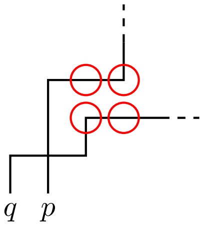
We consider three cases:



1423
13254



1423, 12543
215643, 216543



1423, 12543, 25143
216543, 241653

Conclusion

When w avoids the patterns 1423, 12543, 13254, 25143, 215643, 216543, and 241653 we have the following:

Theorem

Let $\text{co-BPD}(w) = \{\text{co}(B) : B \in \text{BPD}, \text{co}(B)$ ~~*is reduced*~~$\}$ *and for a* co-BPD, C, *let* $\pi(C)$ *be the permutation associated with* C*.*

$$\mathfrak{G}_w = \sum_{C \in \text{co-BPD}(w)} (-1)^{\ell(\pi(C)) - \ell(w)} \cdot \mathfrak{S}_{\pi(C)}.$$

Future Work

Is there anything else interesting one can say about the set of patterns:

- 1423
- 12543
- 13254
- 25143
- 215643
- 216543
- 241653

Thank you for listening!
Happy Birthday to me!