$\top$-avoiding rectangulations, $I(010, 101, 120, 201) \cong I(011, 201)$, and rushed Dyck paths.

Andrei Asinowski *

University of Klagenfurt (Austria)

Michaela Polley **

Dartmouth College (NH)

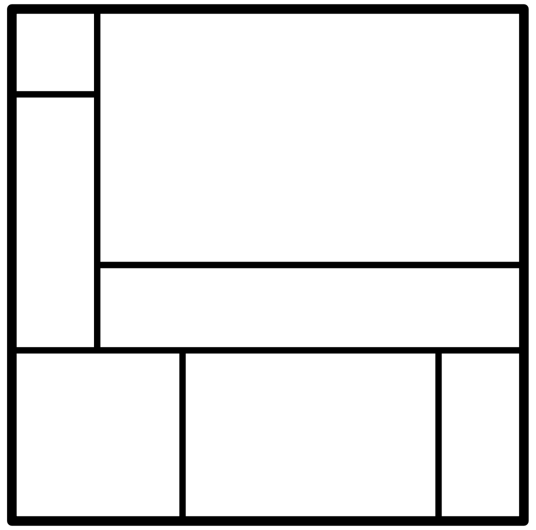
* Supported by the Austrian Science Fund (FWF), [10.55776/P32731].

FWF Austrian Science Fund

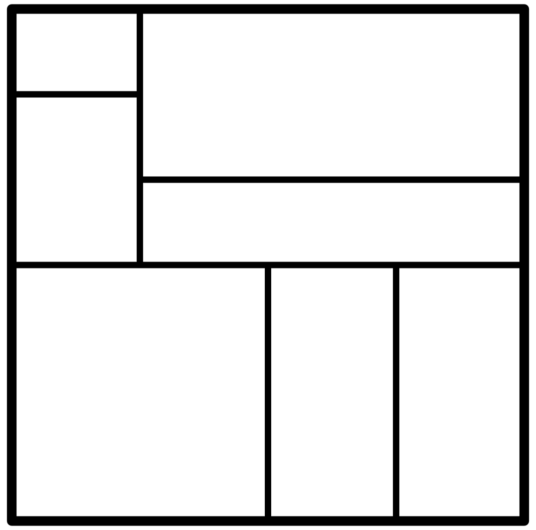
** Supported by the Fulbright-Austrian Marshall Plan Foundation Award.

fulbrightaustria

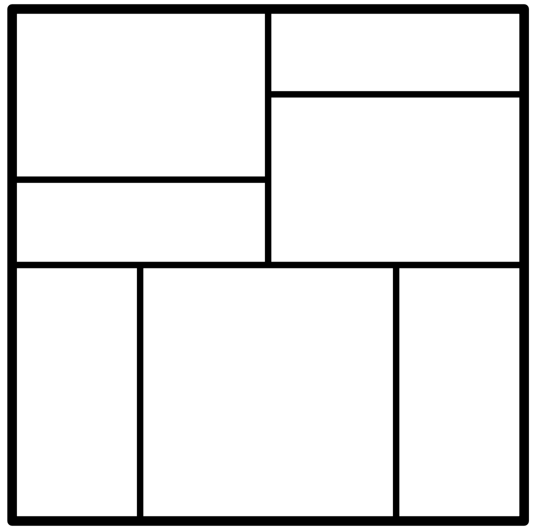
Rectangulations



A



B

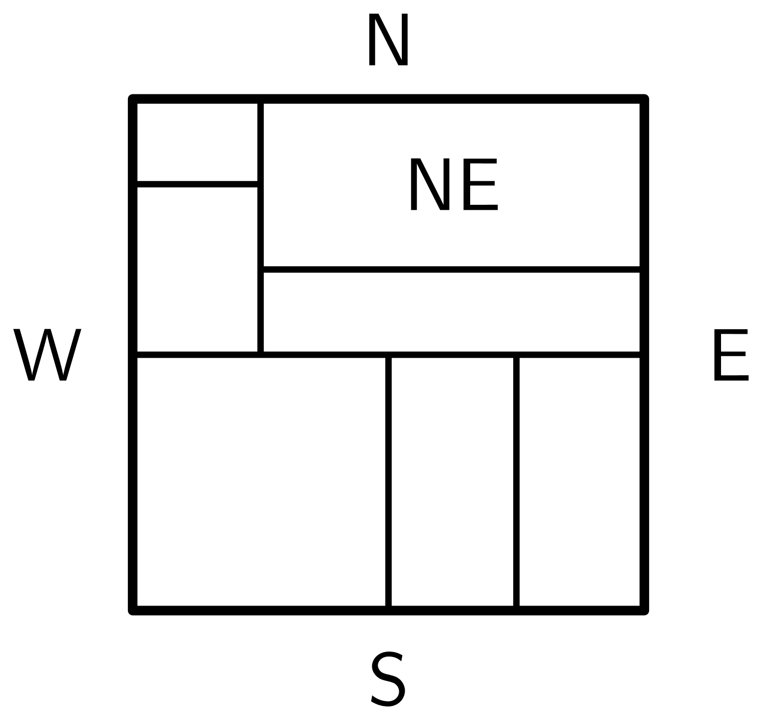


C

All three rectangulations are weakly equivalent, only A and B are strongly equivalent.

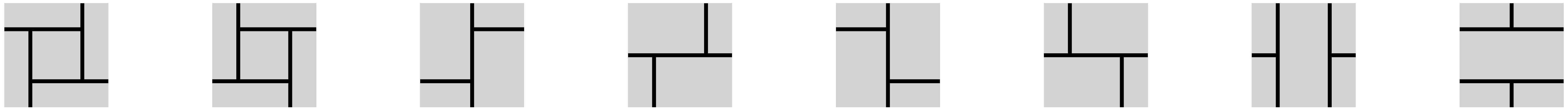
A, B, and C: the same weak rectangulation.
A, B: the same strong rectangulation; C: a different strong rectangulation.

Rectangulations



Patterns in rectangulations

The eight patterns considered by Arturo Merino and Torsten Mütze (2022):



P_1 P_2 P_3 P_4 P_5 P_6 P_7 P_8

- (P_1, P_2)-avoiding rectangulations are precisely guillotine rectangulations.
  (Folklore; Ackerman+Barequet+Pinter, 2006)
- (P_3, P_4)-avoiding strong rectangulations are equinumerous to weak rectangulations.
  (Folklore; Reading, 2012; Cardinal+Sacristán+Silveira, 2018)
- (P_3, P_4, P_5, P_6)-avoiding strong rectangulations are precisely area-universal rectangulations.
  (Eppstein+Mumford+Speckmann+Verbeek, 2012)
- Mesh patterns that correspond to P_1 and P_2 in strong rectangulations.
  (AA+Cardinal+Felsner+Fusy, 2025)

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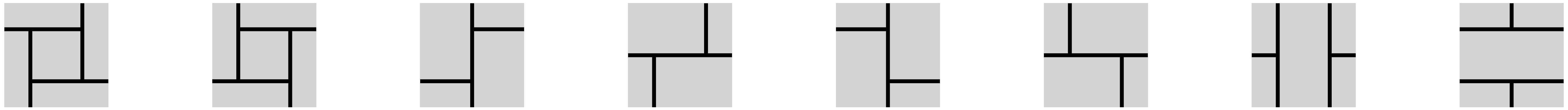

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Merino and Mütze (2023). Combinatorial generation via permutation languages.

- As an application of their algorithm, generated enumerating sequences for all classes of pattern-avoiding rectangulations, where the patterns are from the list given above $\Rightarrow$ some OEIS matches (verified or conjectured).
- Open questions: *The subject of pattern-avoiding rectangulations deserves further systematic investigation, and may still hold many undiscovered gems. [. . .] Does the avoidance of a rectangulation pattern always correspond to the avoidance of a particular permutation pattern, and what is this correspondence?*

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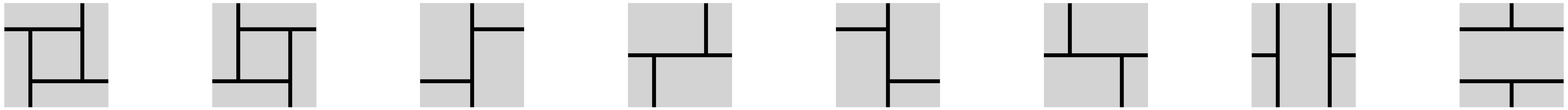

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AA+Banderier (2023): Analytic solution (via permutation patterns) for all the models where all of P_1, P_2, P_3, P_4 and some of P_5, P_6, P_7, P_8 are forbidden.

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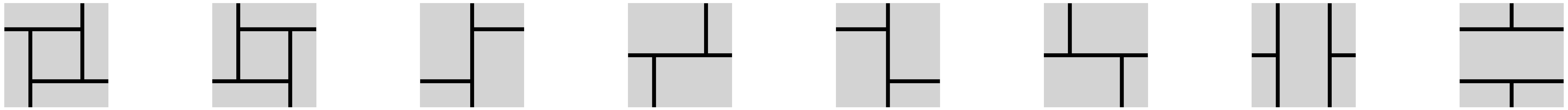

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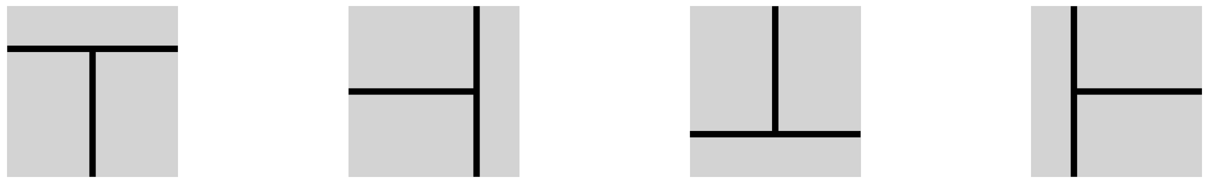
Simpler patterms:



- Aaron Williams (2023): $\top$-avoiding **weak** rectangulations are enumerated by Catalan numbers.
- Torsten Mütze + Namrata (2023) generated the enumerating sequence for $\top$-avoiding **strong** rectangulations and observed that it matches OEIS A279555.
- Our study: All combinations of $\top$-like patterns, in weak and in strong rectagulations.

Enumeration of classes of rectangulations that avoid $\top$-like patterns

Summary of results (presented by Michaela Polley at PP24):

$\mathcal{L} \subseteq \{\top, \vdash, \bot, \dashv\}$	Enumeration of $\mathcal{L}$-avoiding rectangulations
$\top$	weak: Catalan numbers strong: OEIS A279555
$\top$ $\bot$	weak: 2^{n-1} strong: OEIS A287709
$\top$ $\vdash$	2^{n-1}
$\top$ $\vdash$ $\bot$	n
$\top$ $\vdash$ $\bot$ $\dashv$	2

OEIS A279555

A279555 Number of length n inversion sequences avoiding the patterns 110, 210, 120, and 010. 27

1, 1, 2, 5, 15, 51, 189, 746, 3091, 13311, 59146, 269701, 1256820, 5966001, 28773252, 140695923, 696332678, 3483193924, 17589239130, 89575160517, 459648885327, 2374883298183, 12346911196912, 64555427595970, 339276669116222, 1791578092326881, 9501960180835998

(list; graph; refs; listen; history; text; internal format)

OFFSET 0,3

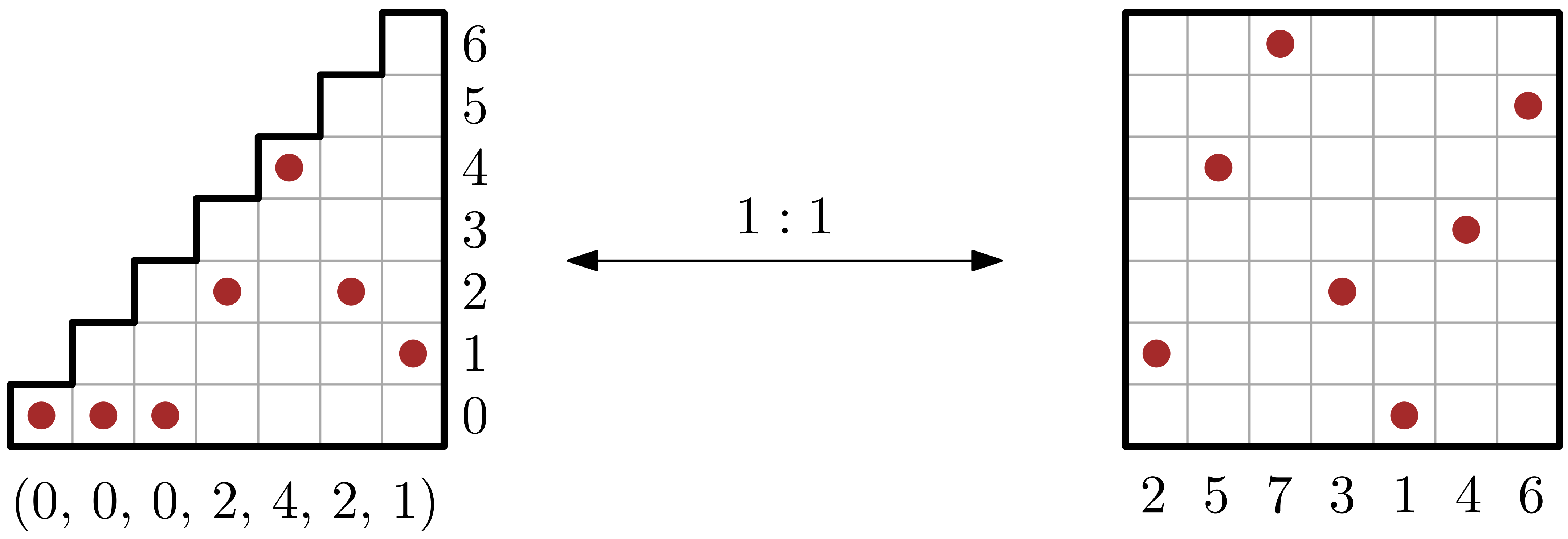
COMMENTS A length n inversion sequence e_1e_2...e_n is a sequence of integers where 0 <= e_i <= i-1. The term a(n) counts those length n inversion sequences with no entries e_i, e_j, e_k (where i<j<k) such that e_j > e_k and e_i >= e_k. This is the same as the set of length n inversion sequences avoiding 010, 110, 120, and 210.

It can be shown that this sequence also counts the length n inversion sequences with no entries e_i, e_j, e_k (where i<j<k) such that e_i <> e_j >=e_k and e_i >= e_k. This is the same as the set of length n inversion sequences avoiding 010, 100, 120, and 210.

LINKS Megan A. Martinez and Carla D. Savage, Patterns in Inversion Sequences II: Inversion Sequences Avoiding Triples of Relations, arXiv:1609.08106 [math.CO], 2016.

Inversion sequences

An *inversion sequence* is an integer sequence $e = (e_1, e_2, \ldots, e_n)$ such that $0 \le e_j \le j - 1$ for each $1 \le j \le n$.



$e = (e_1, e_2, \ldots, e_n)$ is the inversion sequence of $\pi = \pi_1\pi_2 \ldots \pi_n$, that is: $e_j = \left|\{i\colon\ i < j,\ \pi_i > \pi_j\}\right|$.

Patterns: straightforward. (NB: A pattern is not necessarily an inversion seqence.)

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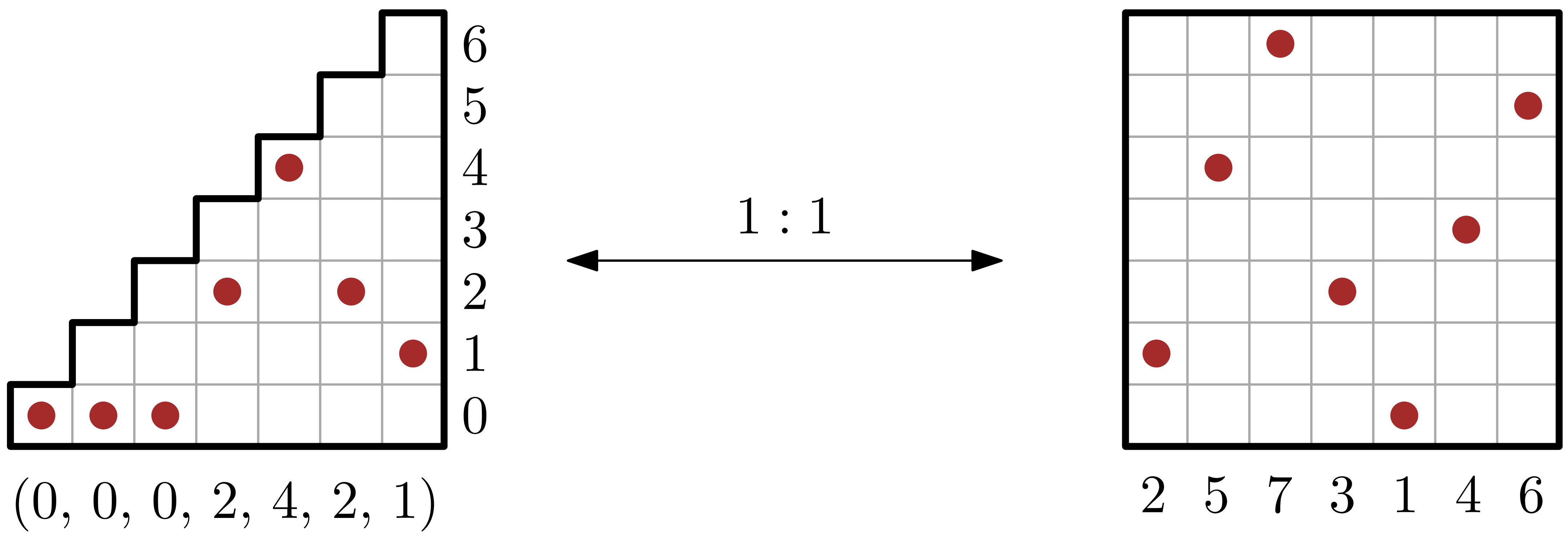

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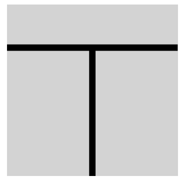
High elements: $\{j\colon\ e_j = j - 1\}$.

⊤-avoiding rectangulations: computations and conjectures



weak: Catalan numbers
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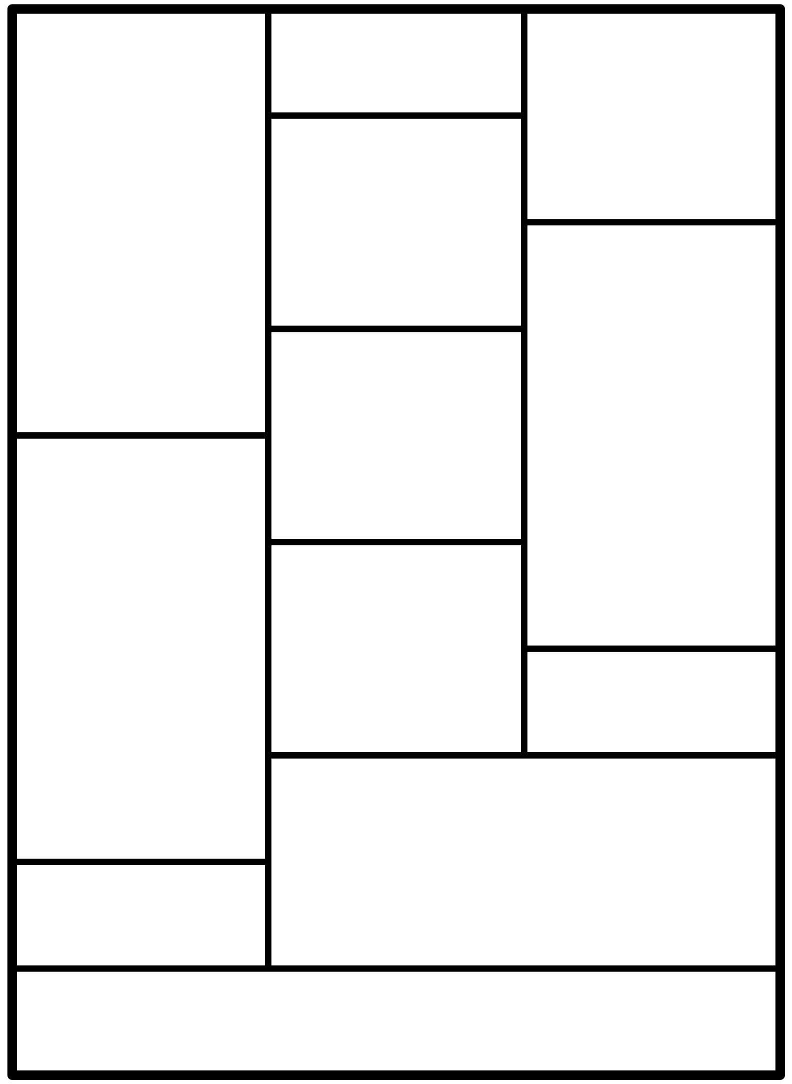

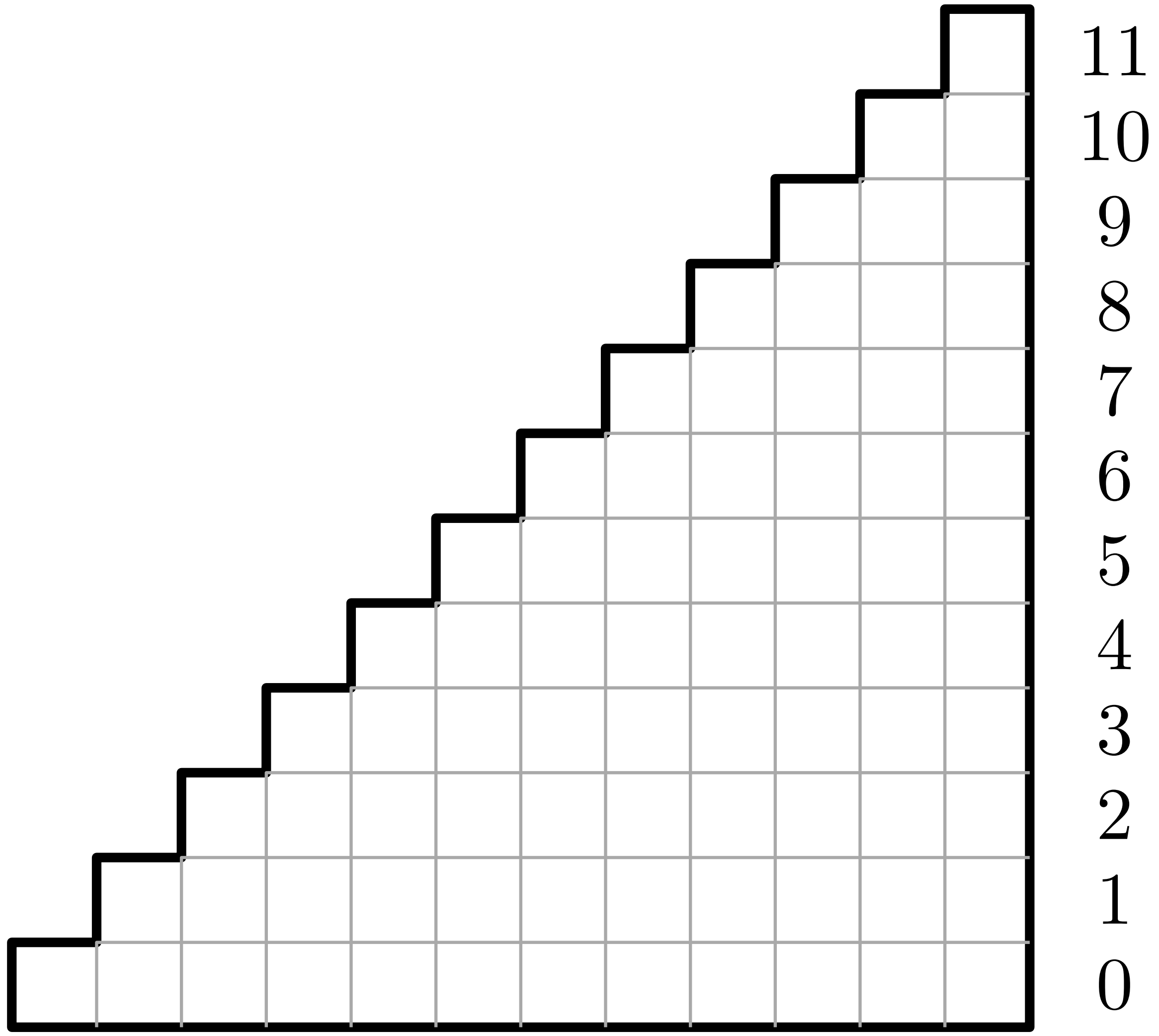

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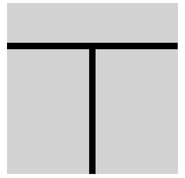

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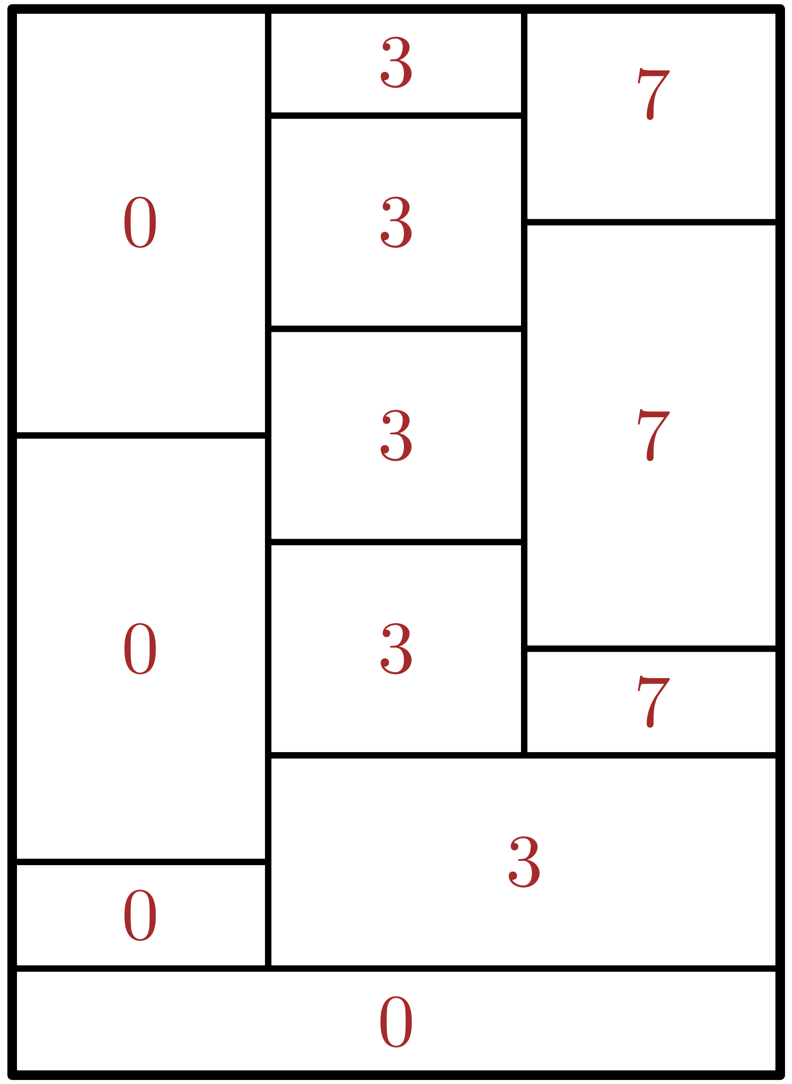
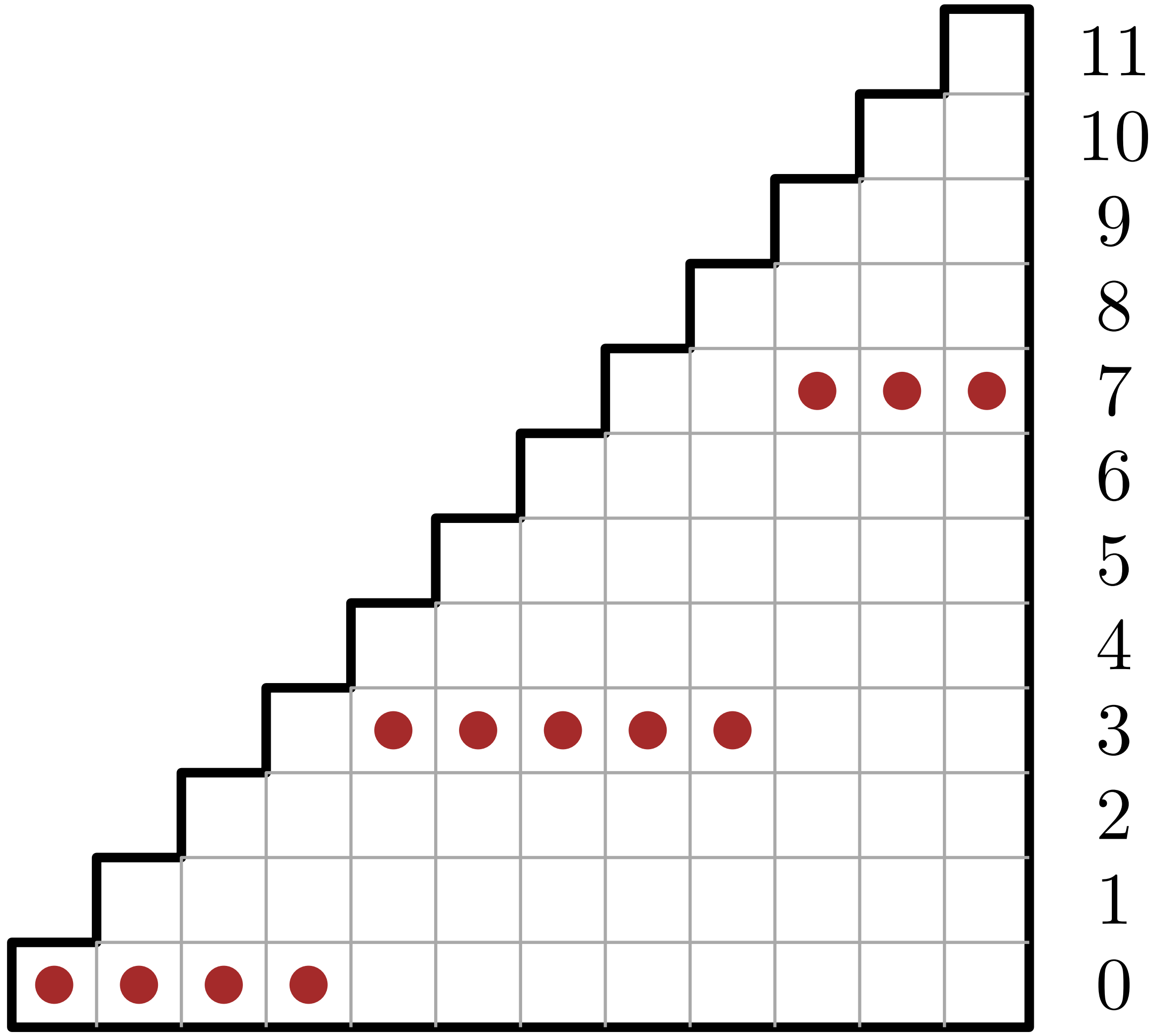
(10)-avoiding inversion sequences





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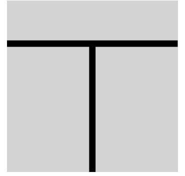
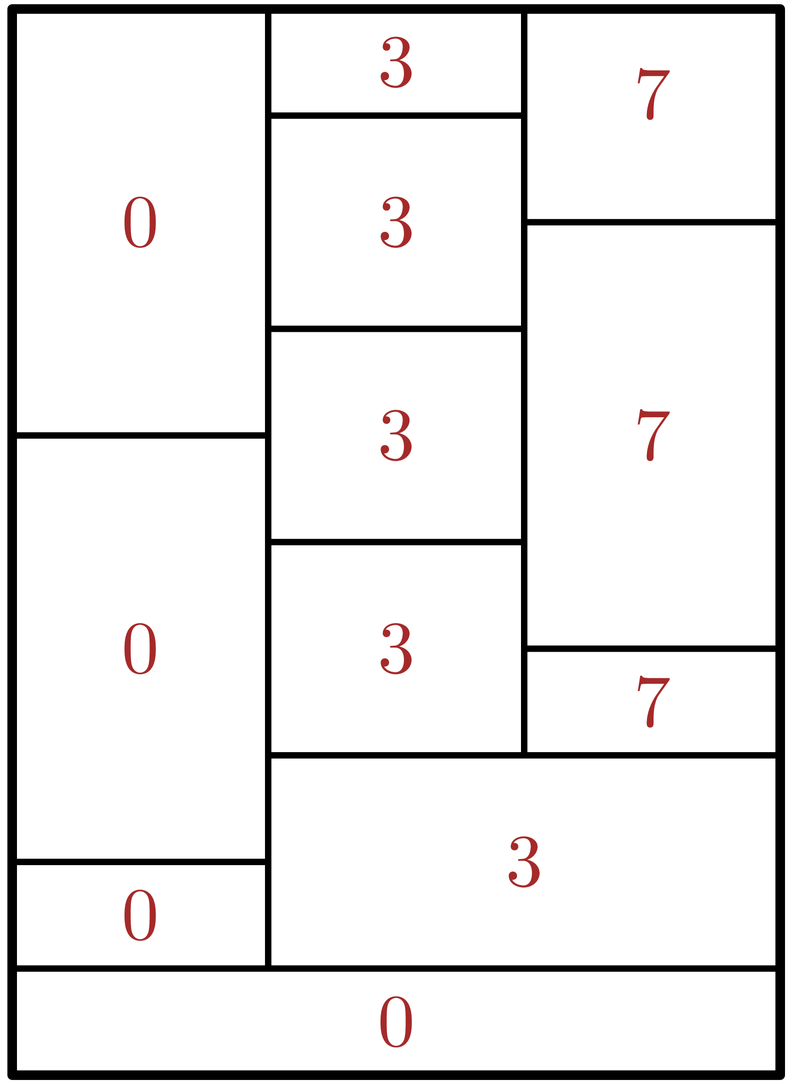
	weak: Catalan numbers ← strong: OEIS A279555



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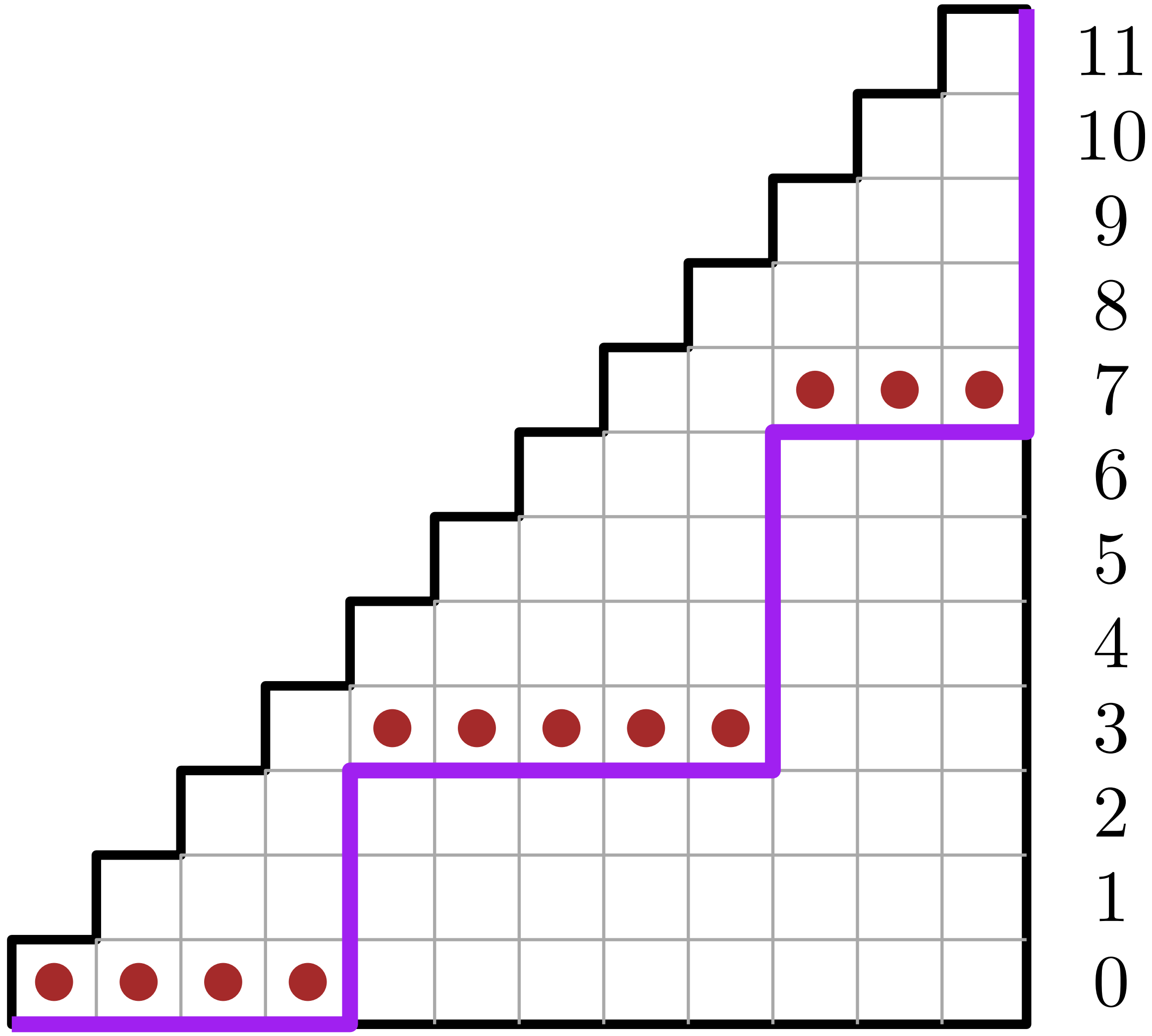

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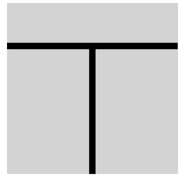
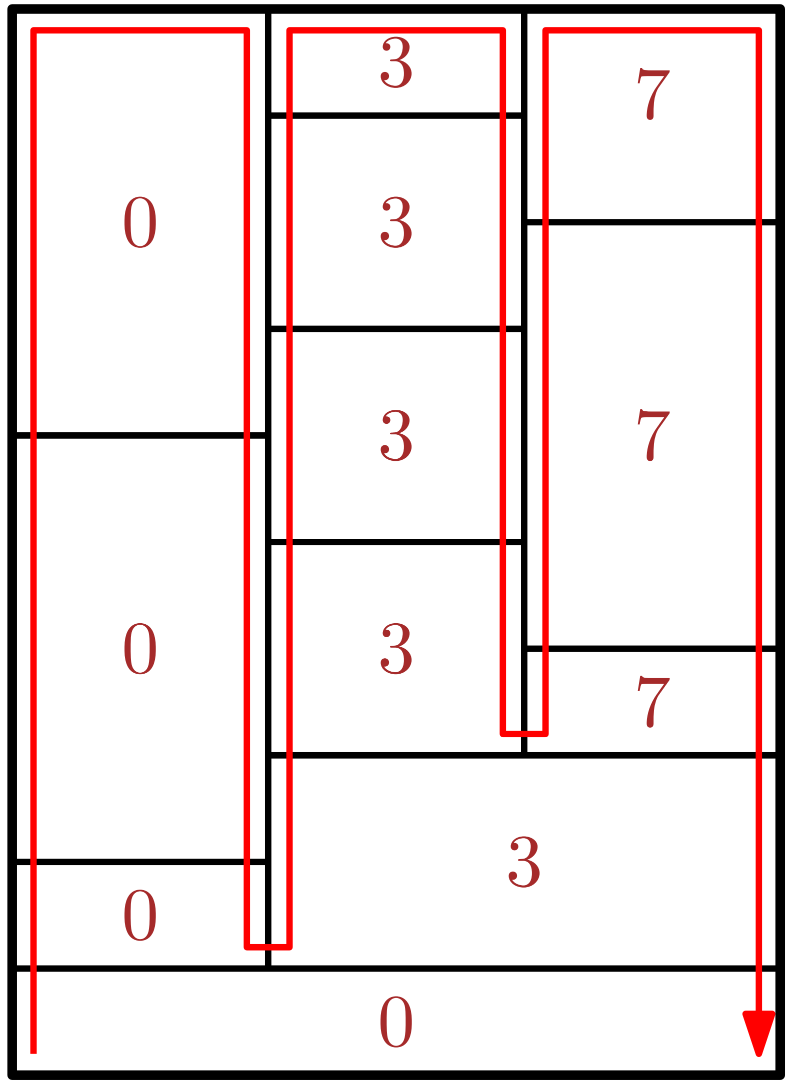
	weak: Catalan numbers strong: OEIS A279555



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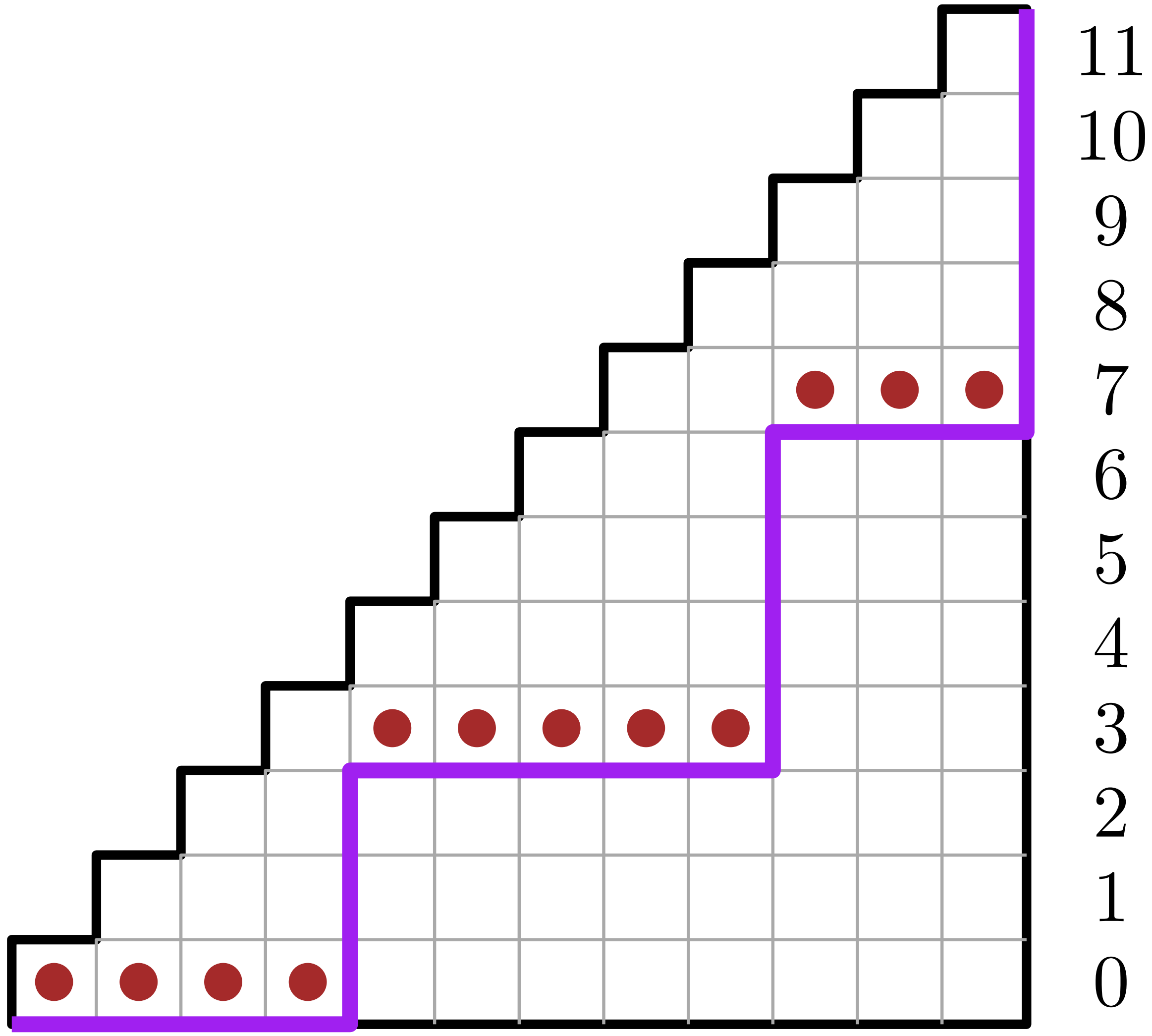

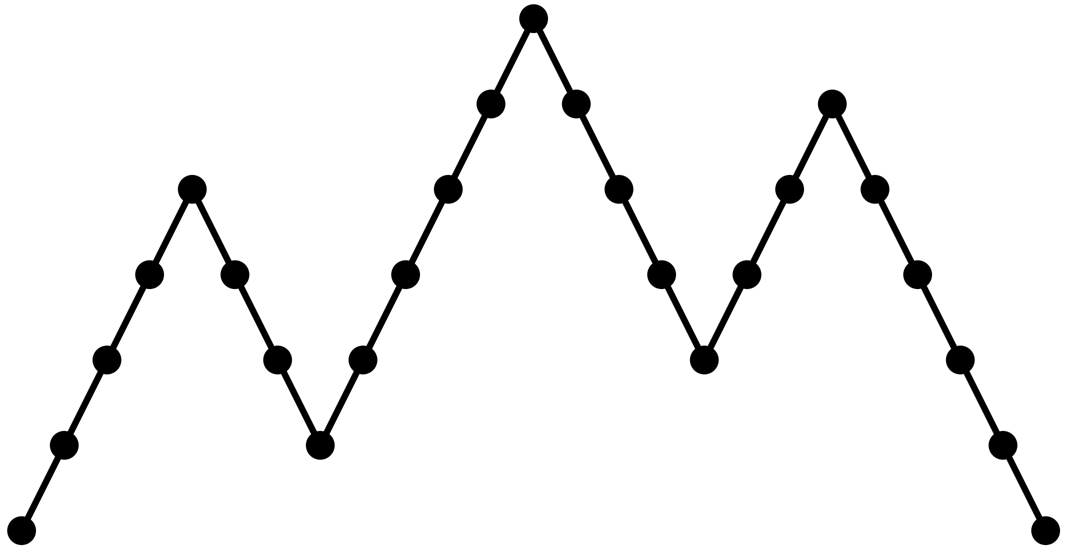

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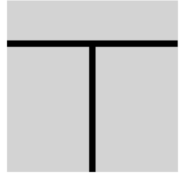	weak: Catalan numbers strong: OEIS A279555

A279555: Number of inversion sequences of length n that avoid the patterns 010, 110, 120, and 210.
Megan Martinez and Carla Savage. Patterns in inversion sequences II (2018).

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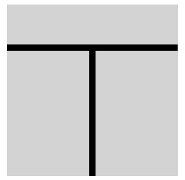	weak: Catalan numbers strong: OEIS A279555

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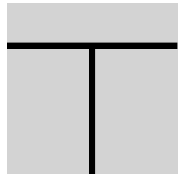

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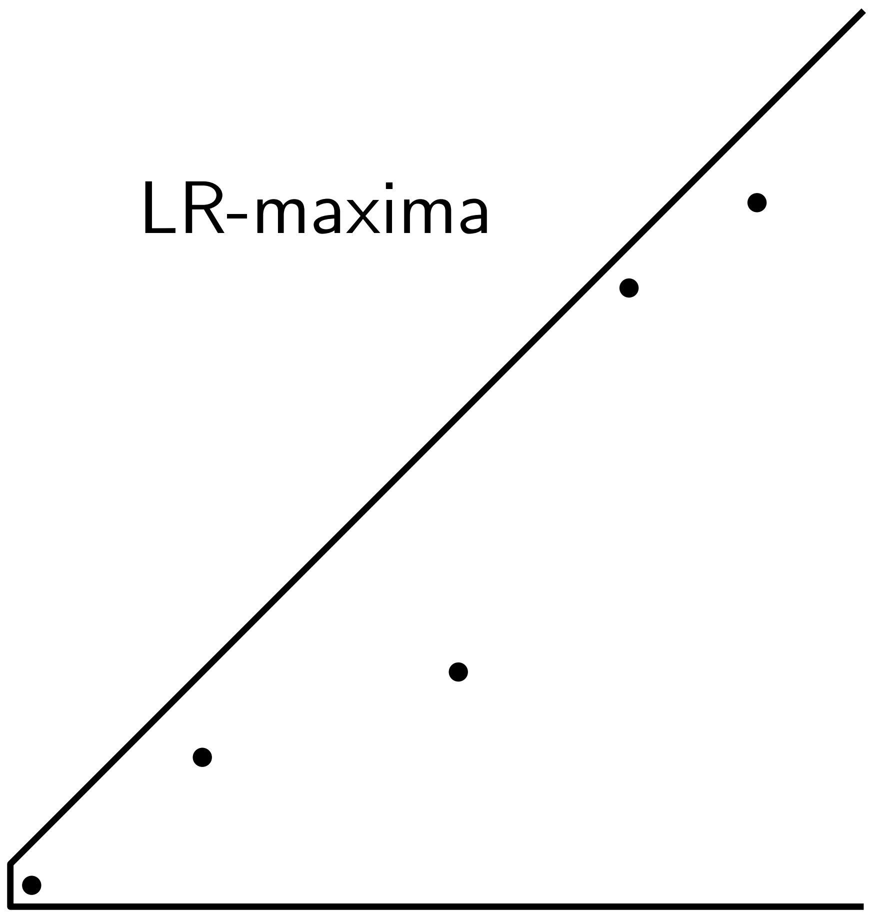

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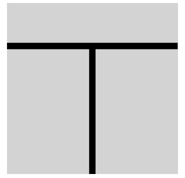	weak: Catalan numbers strong: OEIS A279555

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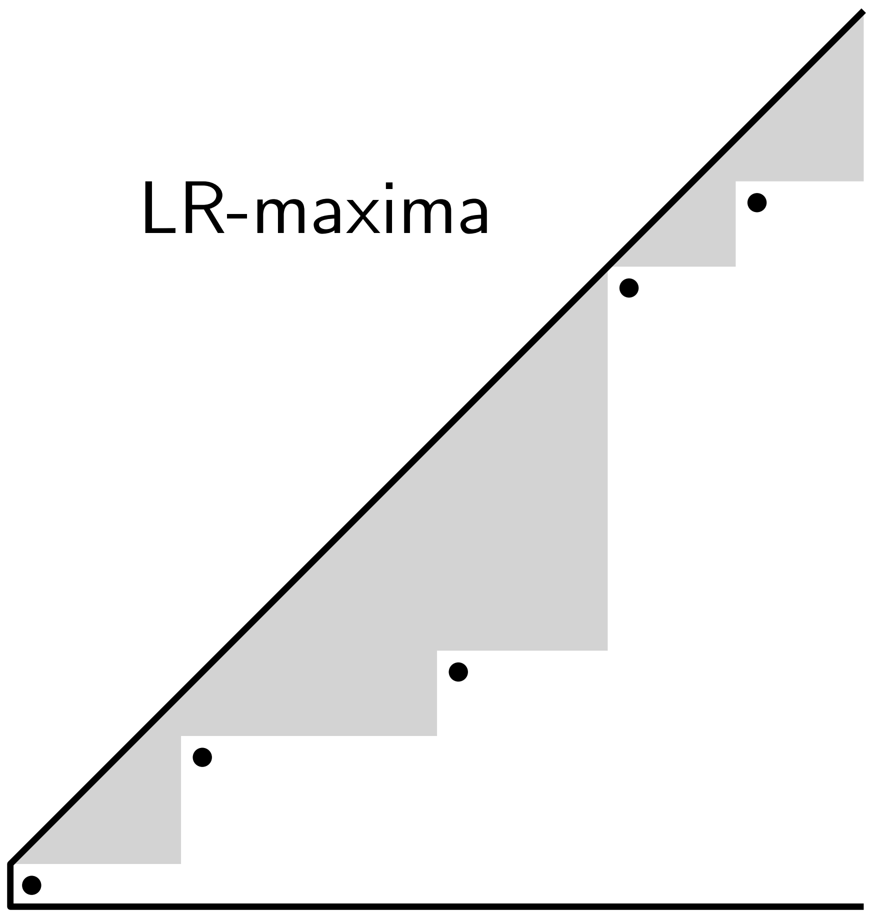

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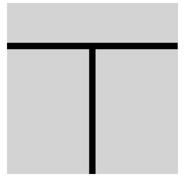

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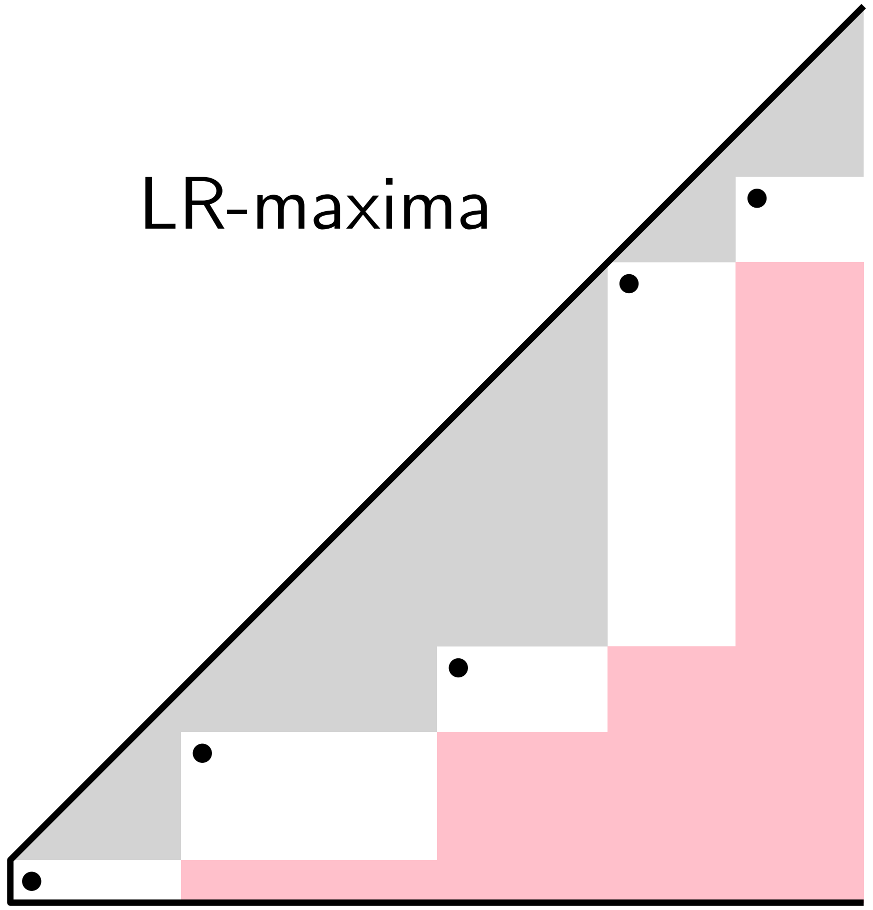

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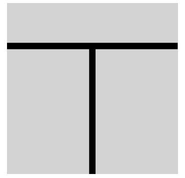	weak: Catalan numbers strong: OEIS A279555

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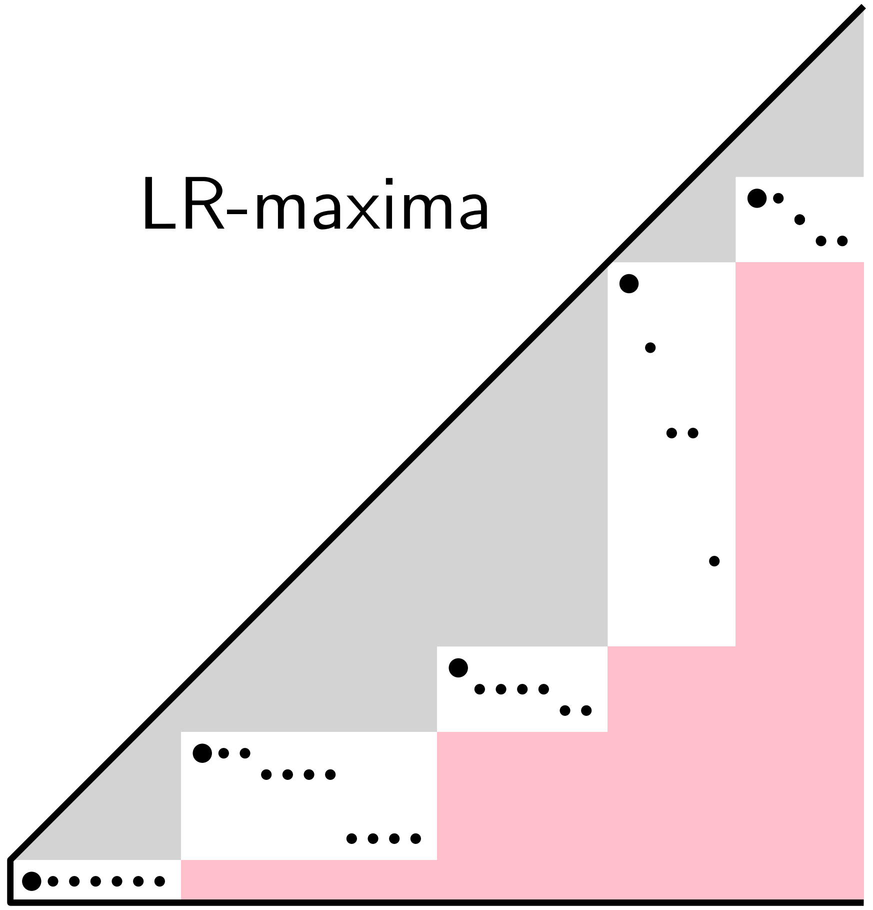

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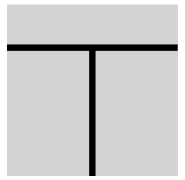	weak: Catalan numbers strong: OEIS A279555

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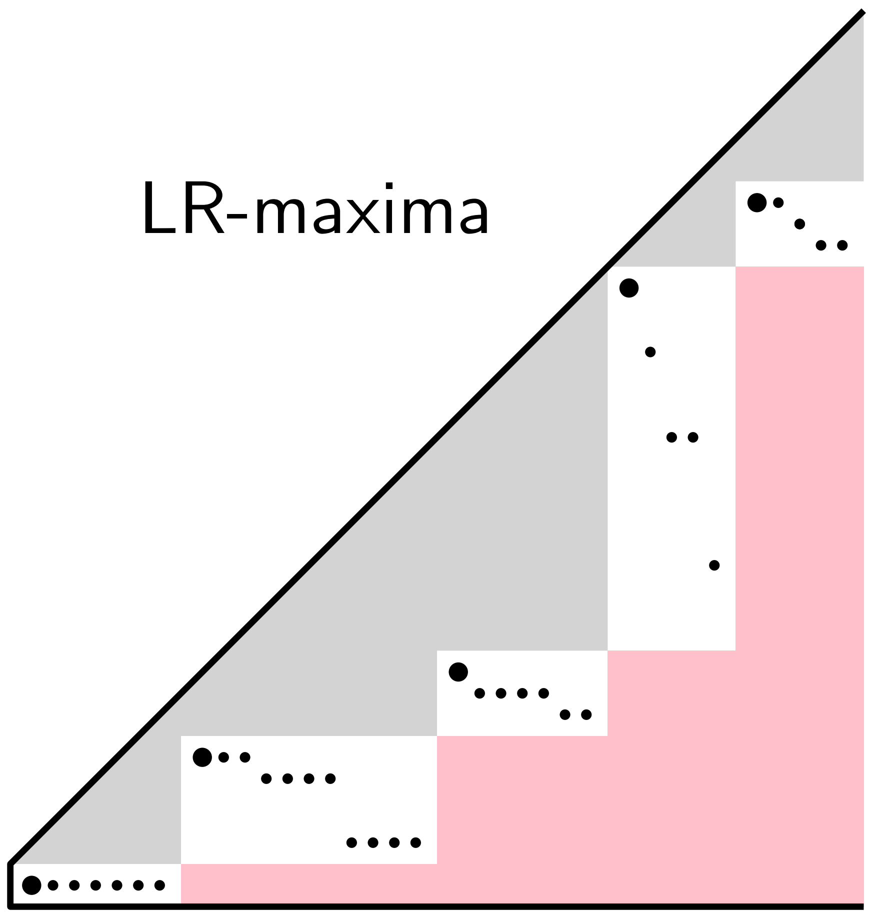

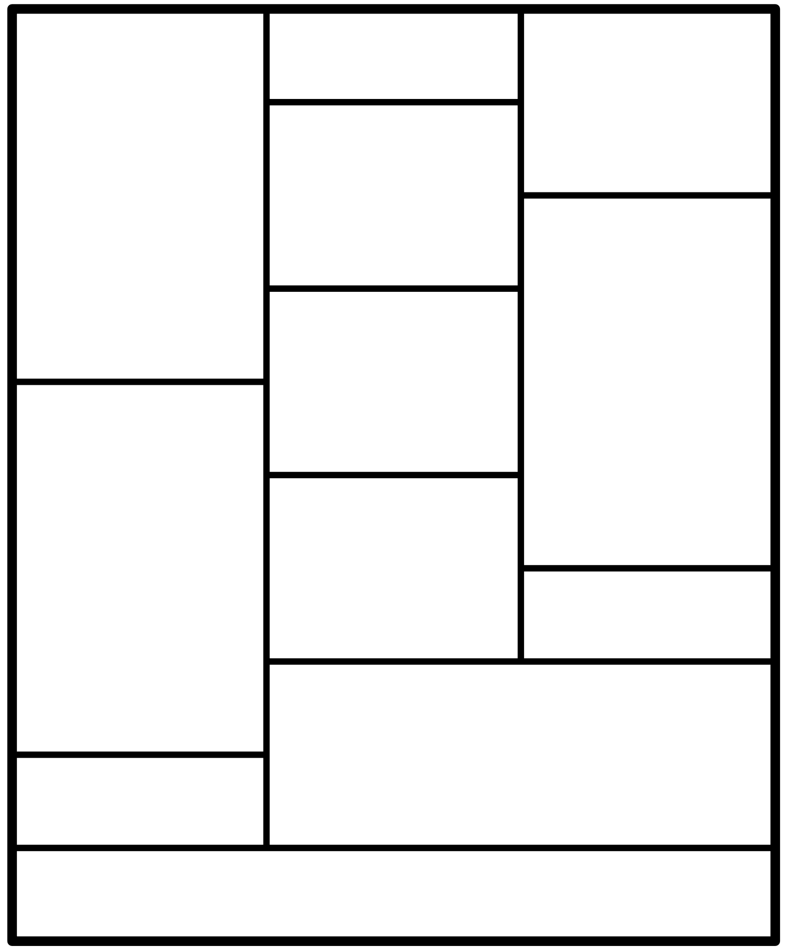

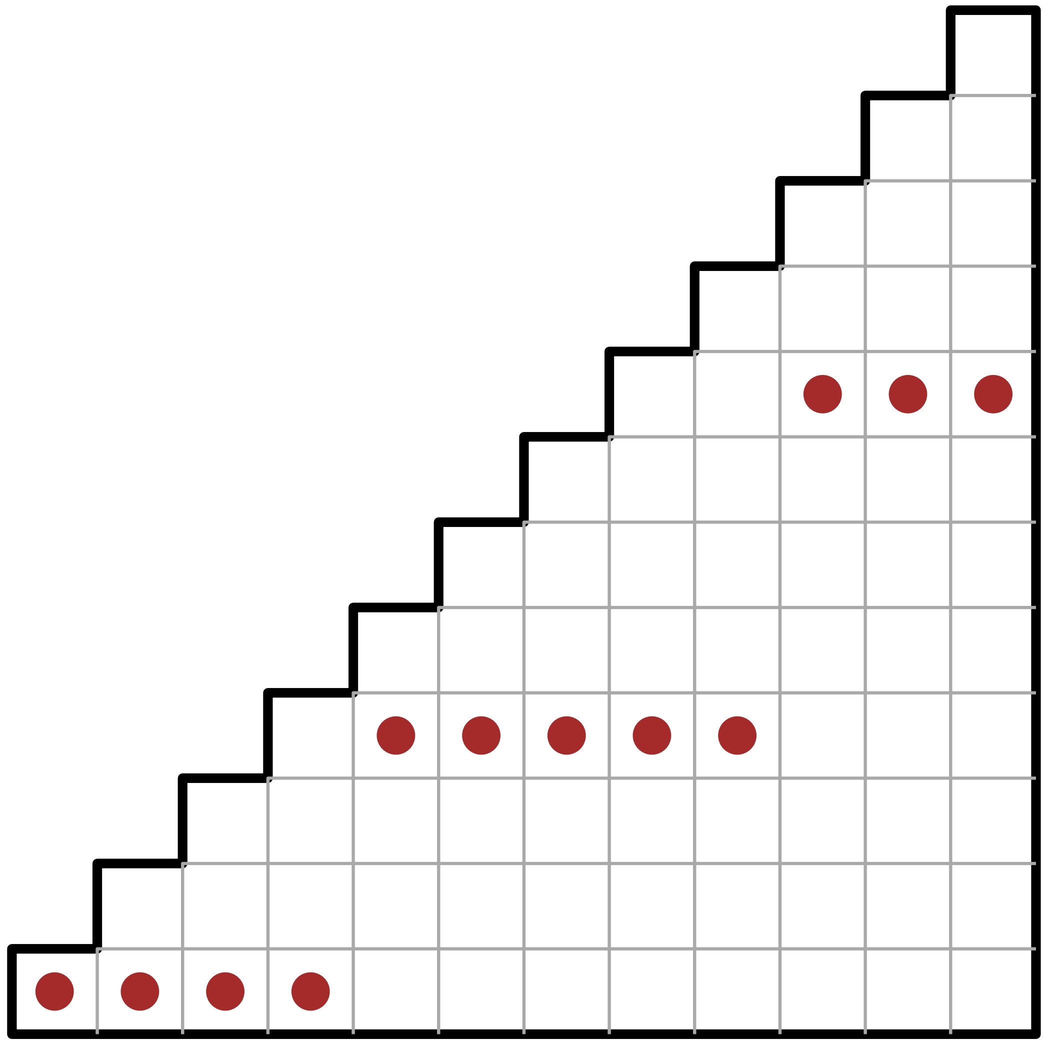

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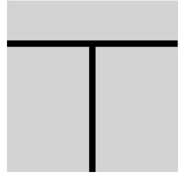	weak: Catalan numbers strong: OEIS A279555

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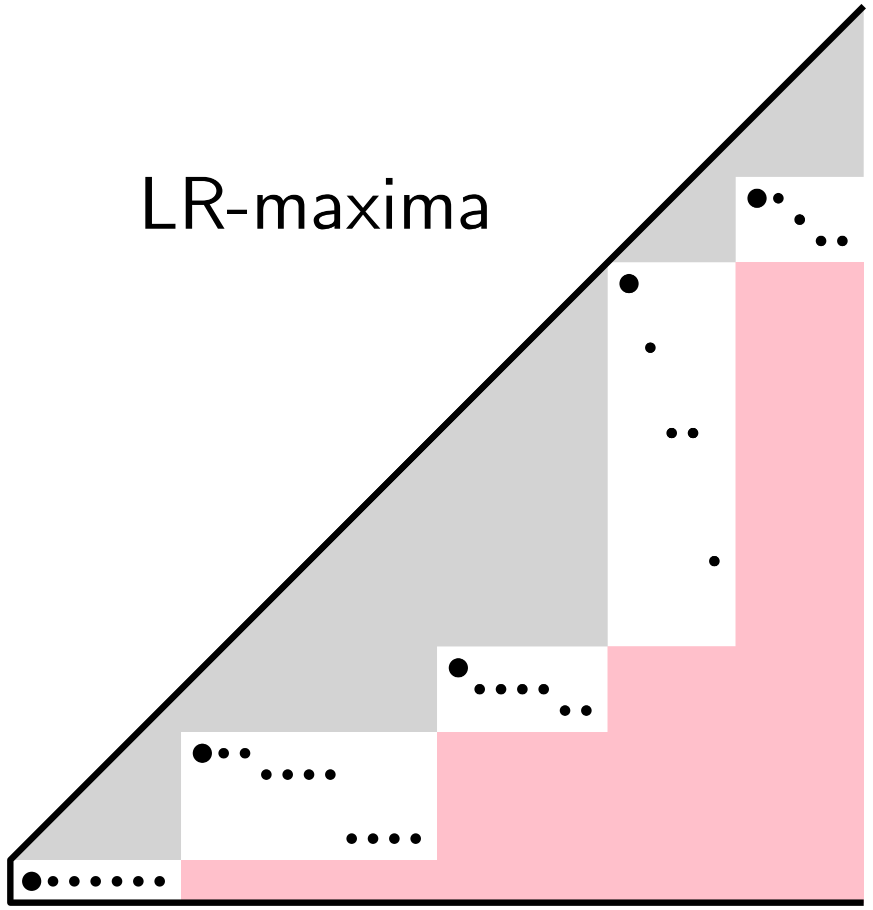

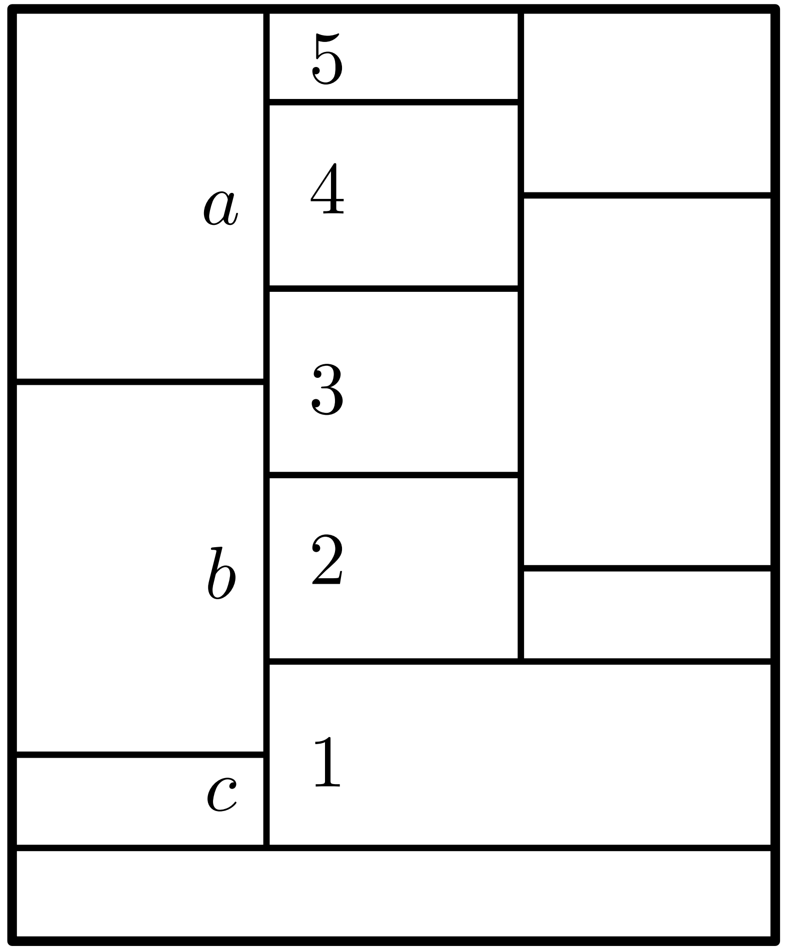

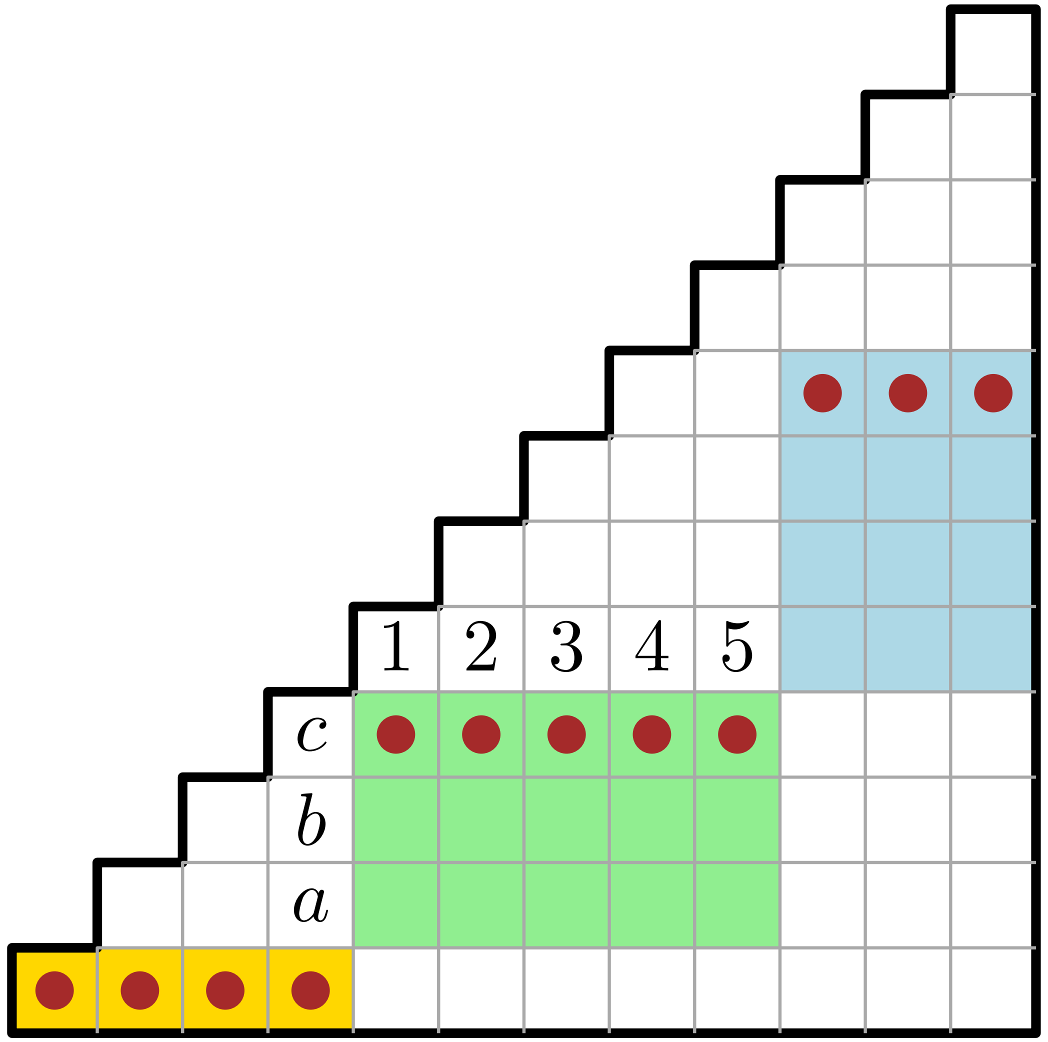

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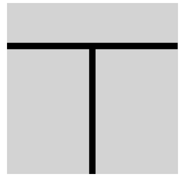	weak: Catalan numbers strong: OEIS A279555 ←

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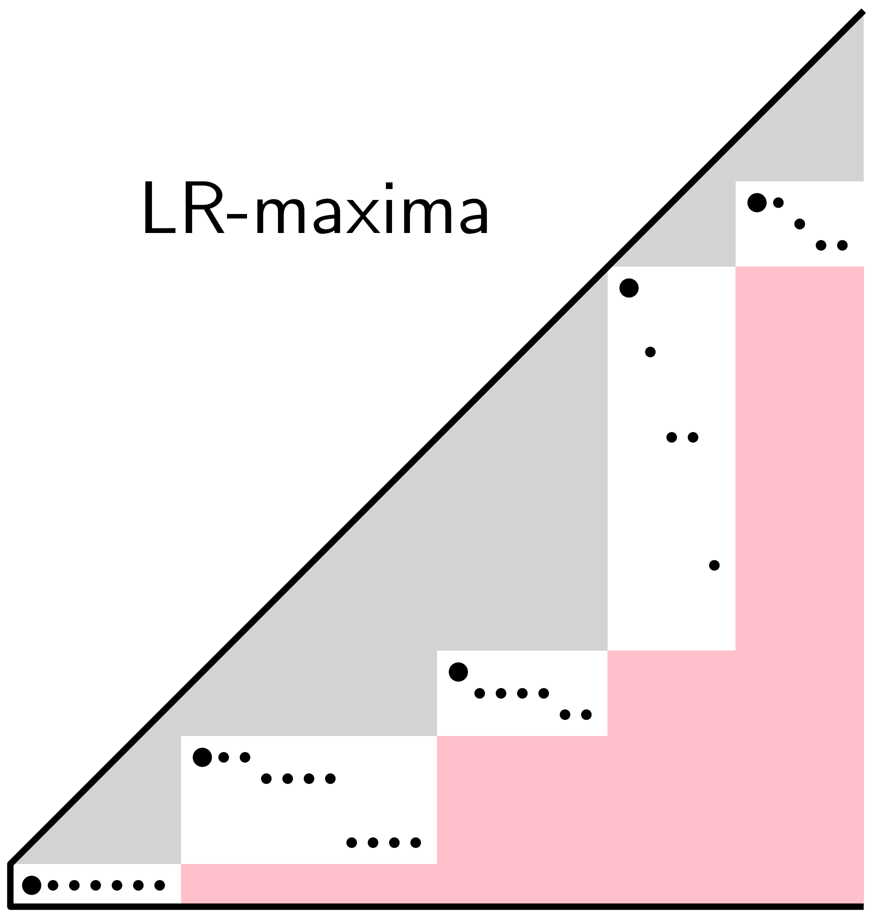

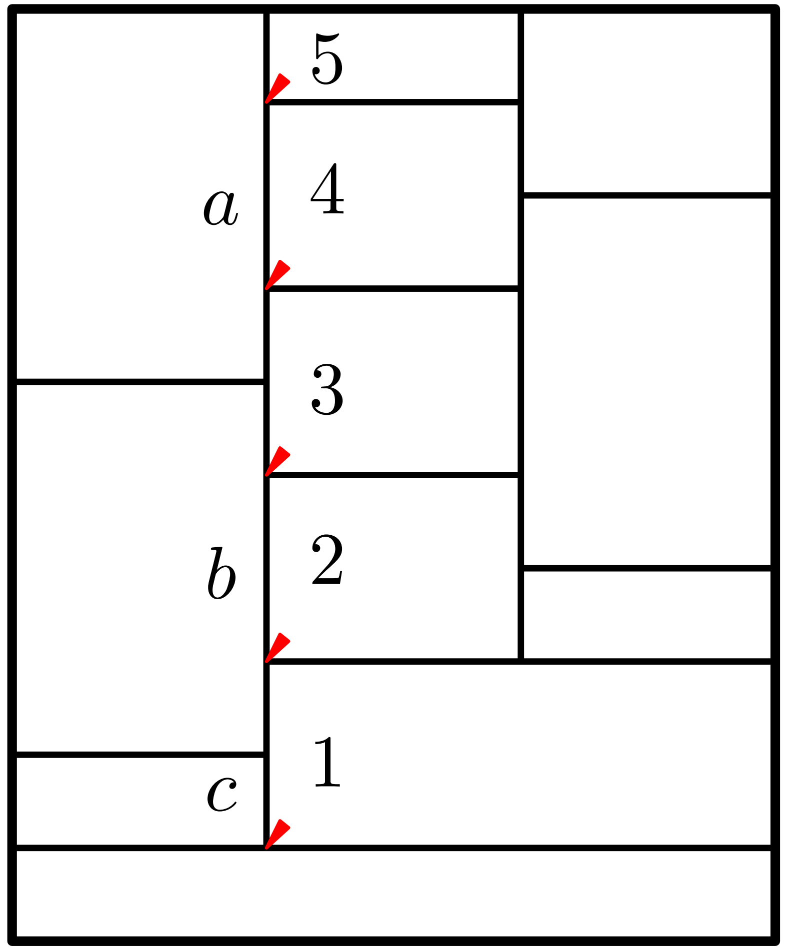

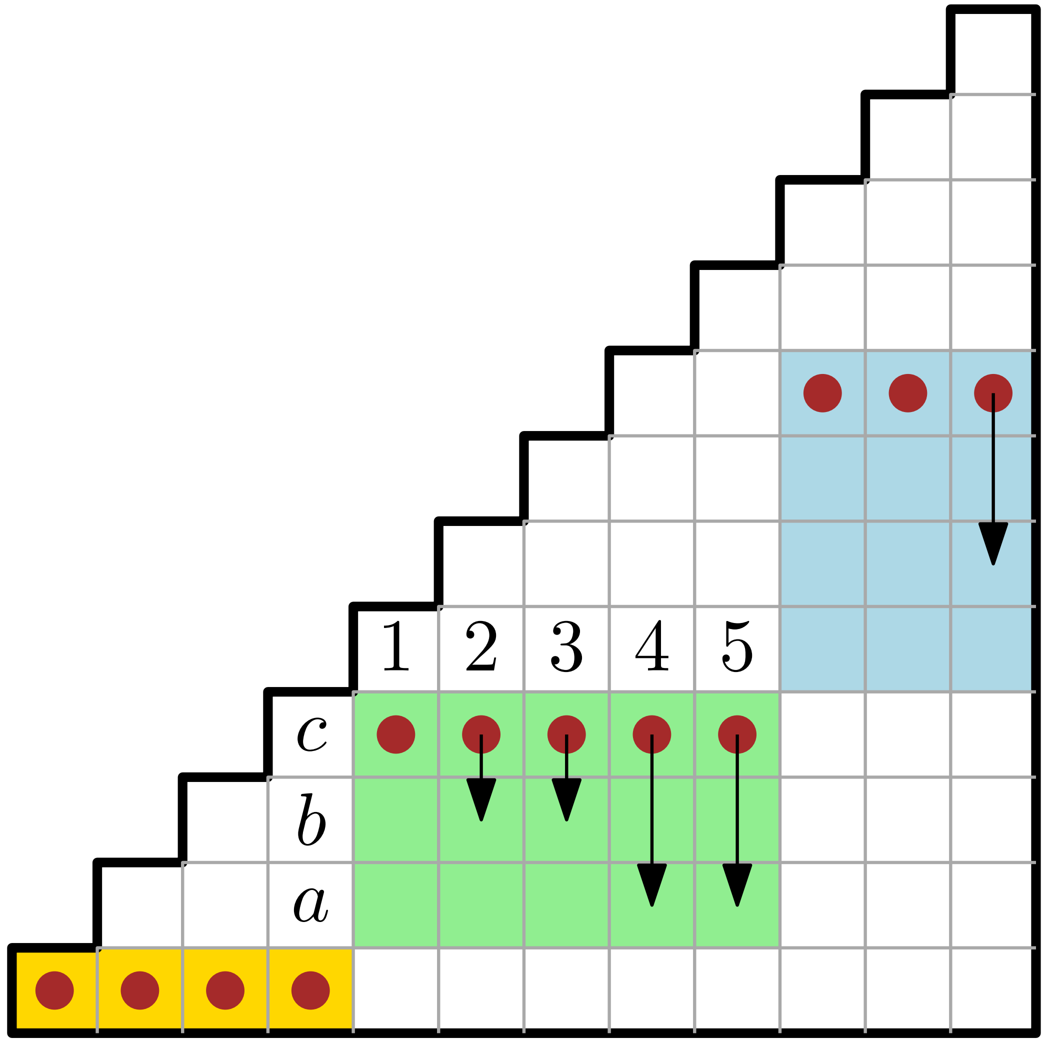

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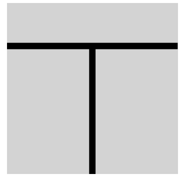

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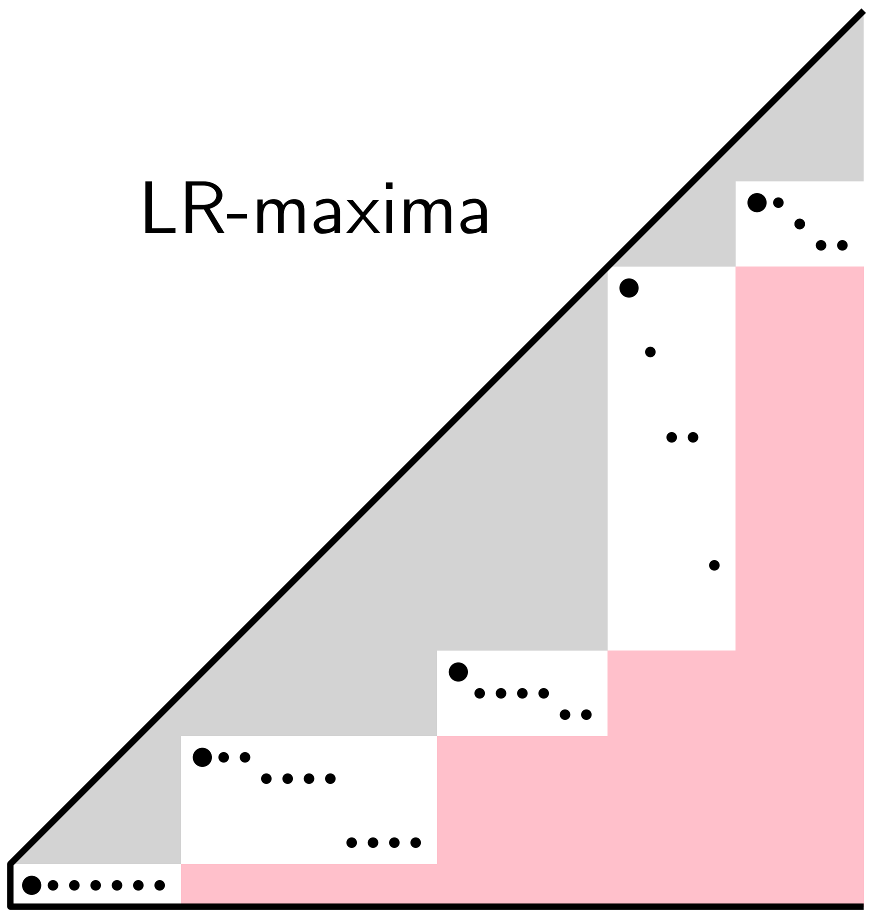
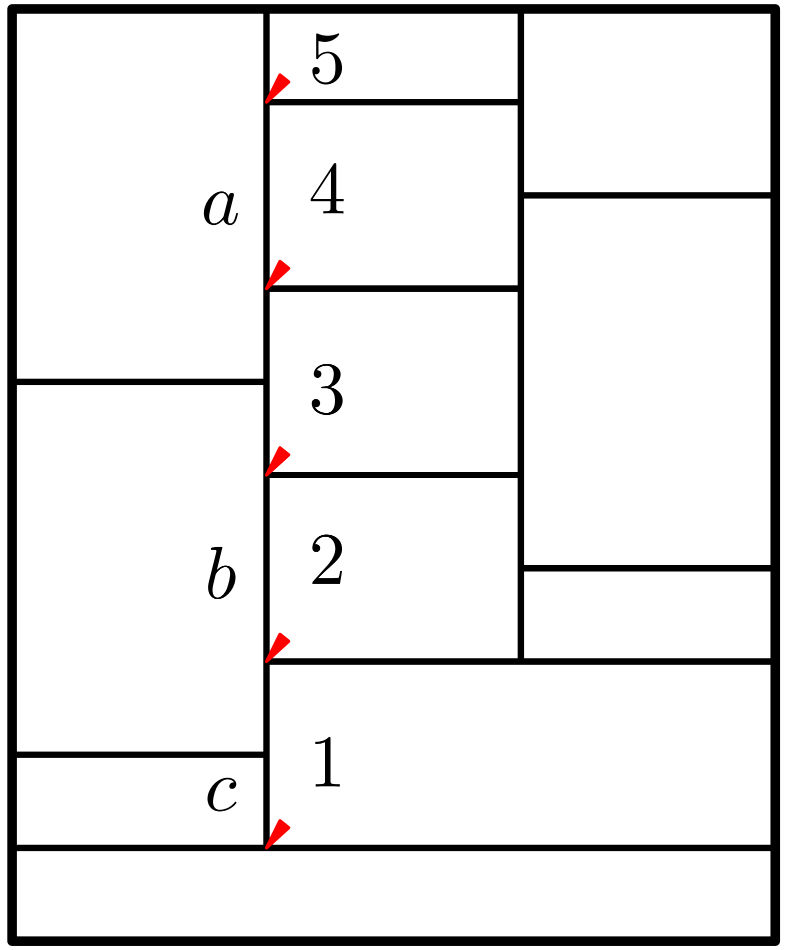
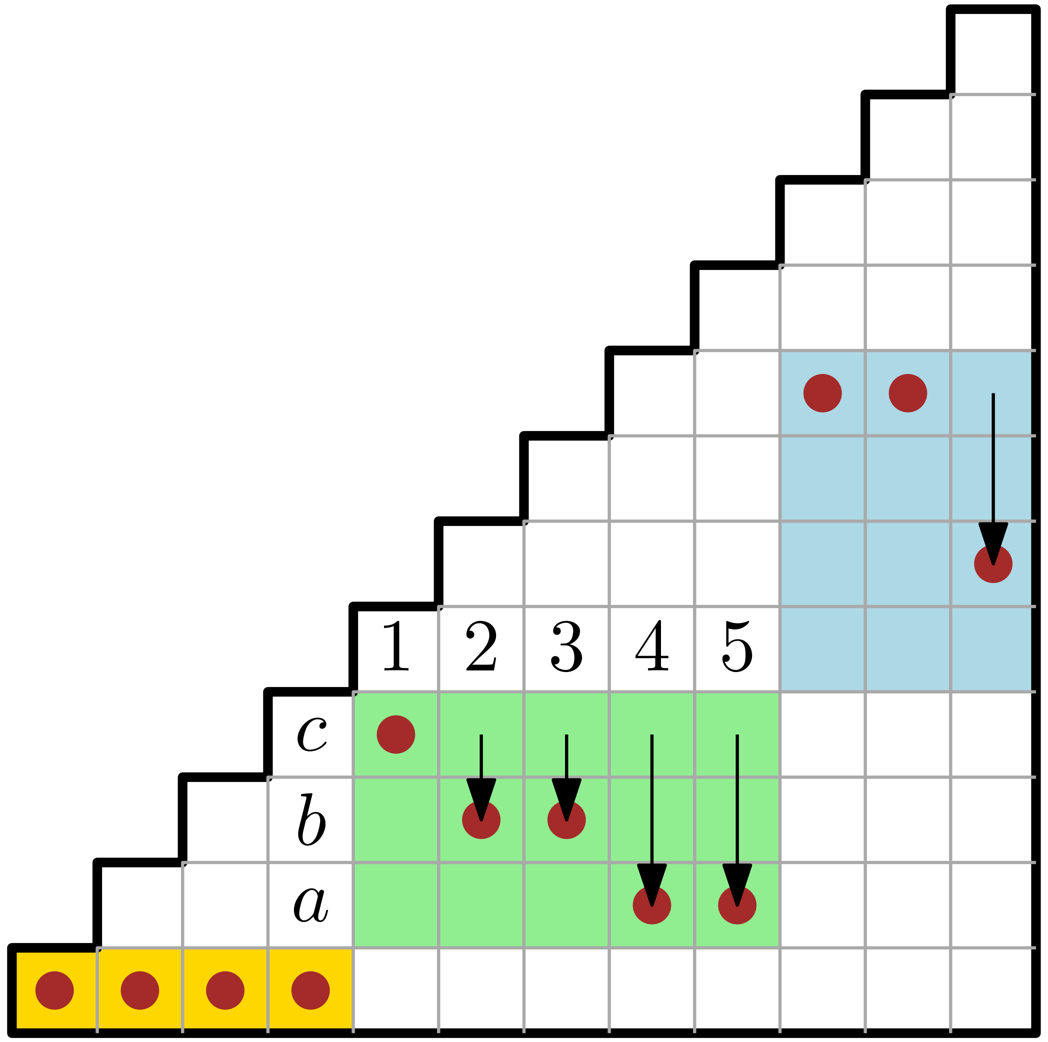
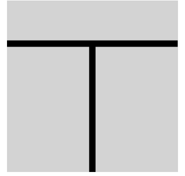




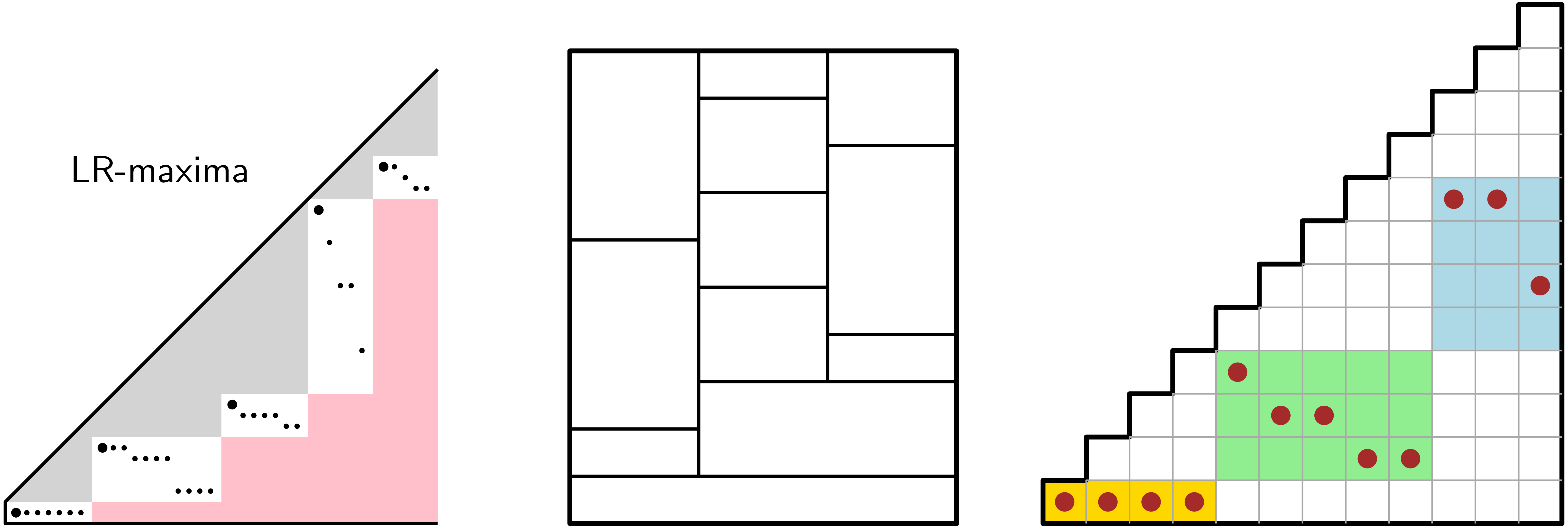


⊤-avoiding rectangulations: computations and conjectures



weak: Catalan numbers
strong: OEIS A279555

010, 101, 120, and 201
A279555: Number of inversion sequences of length n that avoid the patterns ~~010, 110, 120, and 210~~.
Megan Martinez and Carla Savage. Patterns in inversion sequences II (2018).



Classes of inversion sequences enumerated by A279555 $(1, 2, 5, 15, 51, 189, 746, 3091 \ldots)$

AA+MP: $\top$-avoiding strong rectangulations are enumerated by A279555, and are bijective to $I(010, 100, 120, 210)$, $I(010, 101, 120, 201)$, and $I(010, 110, 120, 210)$.

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Chunyan Yan and Zhicong Lin (2020). Inversion sequences avoiding pairs of patterns.

$(010, 021)$	Catalan number C_n	Sec. 2.10	A000108	1430,B
$(011, 201)$	$\lvert\mathbf{I}_n(-,>,\geq)\rvert = \lvert\mathbf{I}_n(\neq,\geq,\geq)\rvert$	Open	A279555	3091,A
$(011, 210)$	$\lvert\mathbf{I}_n(-,>,\geq)\rvert = \lvert\mathbf{I}_n(\neq,\geq,\geq)\rvert$	Open	A279555	3091,B

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David Callan and Toufik Mansour (2023). Inversion sequences avoiding quadruple length-3 patterns.

166	{010,100,110,201}		
	{010,100,110,210}		
	{011,101,110,201}	$\overset{r}{\sim}$ $\{011, 210\}$ See [16]	
	{011,101,110,210}	$\overset{r}{\sim}$ $\{011, 210\}$ See [16]	
	{010,101,110,201}		166(1) $\overset{\mathbf{I}}{\sim}$ 166(6) still open
	{010,100,120,210}		
	{010,101,120,201}		
	{010,110,120,210}		166(7) $\overset{\mathbf{I}}{\sim}$ 166(9) holds by Subsection 4.8
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Classes of inversion sequences enumerated by A279555 $(1, 2, 5, 15, 51, 189, 746, \ldots)$

Jay Pantone (2024). The enumeration of inversion sequences avoiding the patterns 201 and 210.

Let $F(x, u, v)$ be the generating function such that the coefficient of $x^n u^k v^\ell$ is the number of 011, 201-avoiding inversion sequences with length n represented by the state (k, ℓ) as above. The succession rules can be converted into the functional equation

$$F(x, u, v) = 1 + xu\left(F(x, u, 1) + \frac{F(x, u, v) - F(x, u, 1)}{v - 1} + \frac{F(x, u, v) - F(x, v, v)}{u - v}\right).$$

Using these succession rules we have calculated the first 500 terms in the counting sequence. They match the 27 given terms of the OEIS sequence A279555, which is defined to be the counting sequence of $\mathbf{I}(010, 110, 120, 210)$. Martinez and Savage [18] prove that this class is equinumerous to $\mathbf{I}(010, 100, 120, 210)$, and Yan and Lin [24] conjecture that both are equinumerous to the class $\mathbf{I}(011, 201)$ for which we have just presented succession rules.[††] As far as we know, this conjecture remains open. We find further evidence for the conjecture by using the succession rules below to generate 500 terms for $\mathbf{I}(010, 100, 120, 210)$ and confirming that they match the counting sequence for $\mathbf{I}(011, 201)$:

$$\begin{aligned}(k, \ell) \longrightarrow\ & (k+1, \ell) \\ & (k+1, i), \quad i \in \{0, \ldots, \ell - 1\} \\ & (i, k - i), \quad i \in \{1, \ldots, k\}.\end{aligned}$$

We omit the details of these succession rules, but their derivation is similar to those for $\mathbf{I}(011, 201)$. They lead to a similar function equation

$$G(x, u, v) = 1 + xu\left(G(x, u, v) + \frac{G(x, u, v) - G(x, u, 1)}{v - 1} + \frac{G(x, u, 1) - G(x, v, 1)}{u - v}\right).$$

Note the full three-variable solutions $F(x, u, v)$ and $G(x, u, v)$ to the above two functional equations are not equal, but the conjecture implies that $F(x, 1, 1) = G(x, 1, 1)$. We do not know if there is a simple way to show this directly from the two functional equations.

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Generating trees for $I(010, 101, 120, 201)$ and for $I(011, 201)$ (Jay Pantone)

Generating tree T1 for $I(010, 101, 120, 201)$:

$$\begin{array}{rlll} \text{Root}: & (1,0). & & \\ \text{Succession rules}: & (k,\ell) & \longrightarrow & (1,k-1),\ (2,k-2),\ \ldots,\ (k,0); \qquad (*) \\ & & & (k+1,\ell),\ (k+1,\ell-1),\ \ldots,\ (k+1,0). \qquad (**) \end{array}$$

Generating tree T2 for $I(011, 201)$:

$$\begin{array}{rlll} \text{Root}: & (1,0). & & \\ \text{Succession rules}: & (k,\ell) & \longrightarrow & (1,k+\ell-1),\ (2,k+\ell-2),\ \ldots,\ (k,\ell); \qquad (*) \\ & & & (k+1,\ell-1),\ (k+1,\ell-2),\ \ldots,\ (k+1,0); \qquad (**) \\ & & & (k+1,0). \qquad (***) \end{array}$$

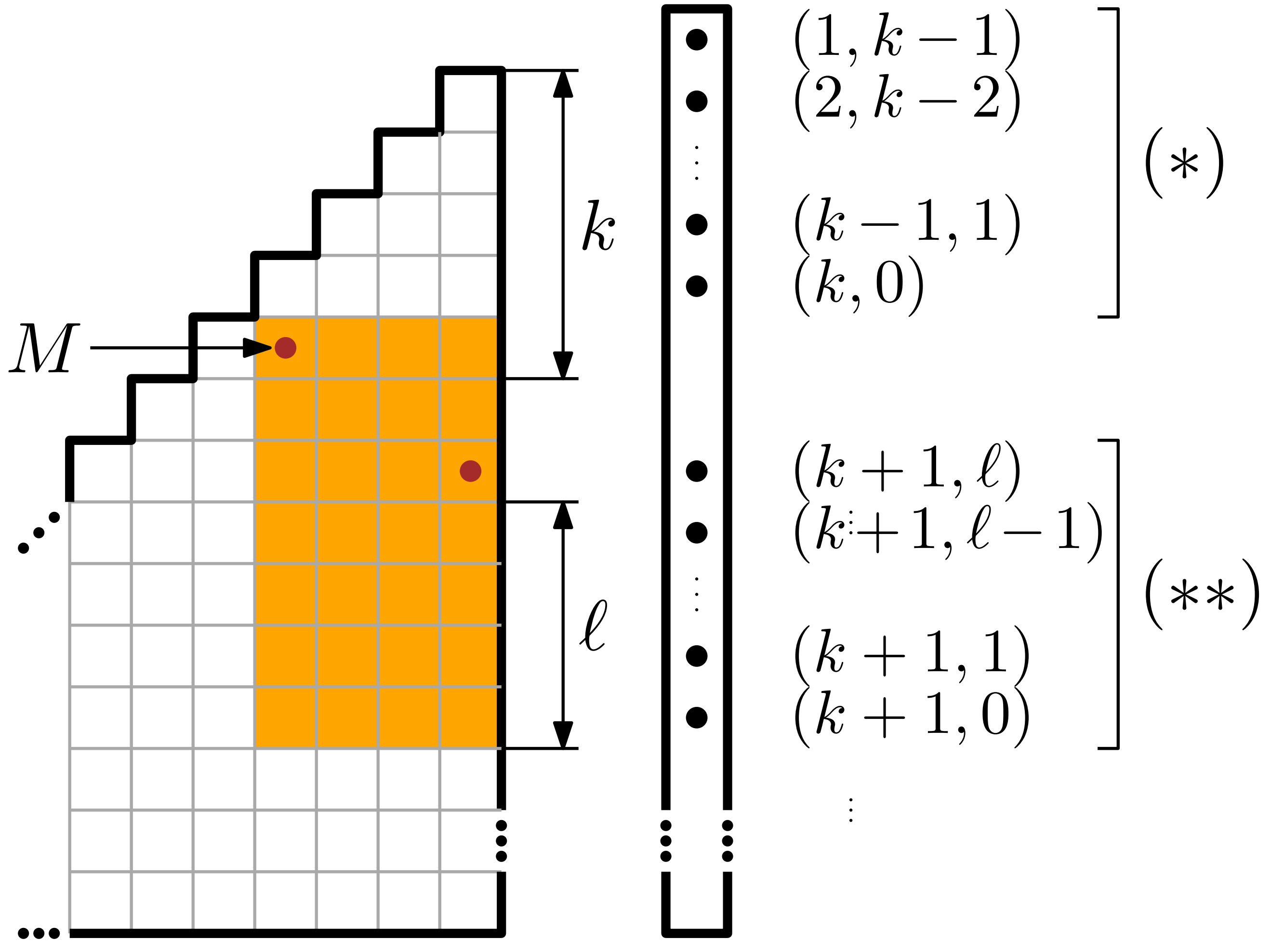
Here, k is the *bounce* defined as $n - M$, where n is the size and M is the maximum value;
ℓ in T1 is the number of admissible values j such that $0 < j < e_n$,
ℓ in T2 is the number of admissible values j such that $0 < j < M$.

T1 is a generating tree for $I(010, 101, 120, 201)$ and for ⊤-avoiding rectangulations

Root : $(1, 0)$.

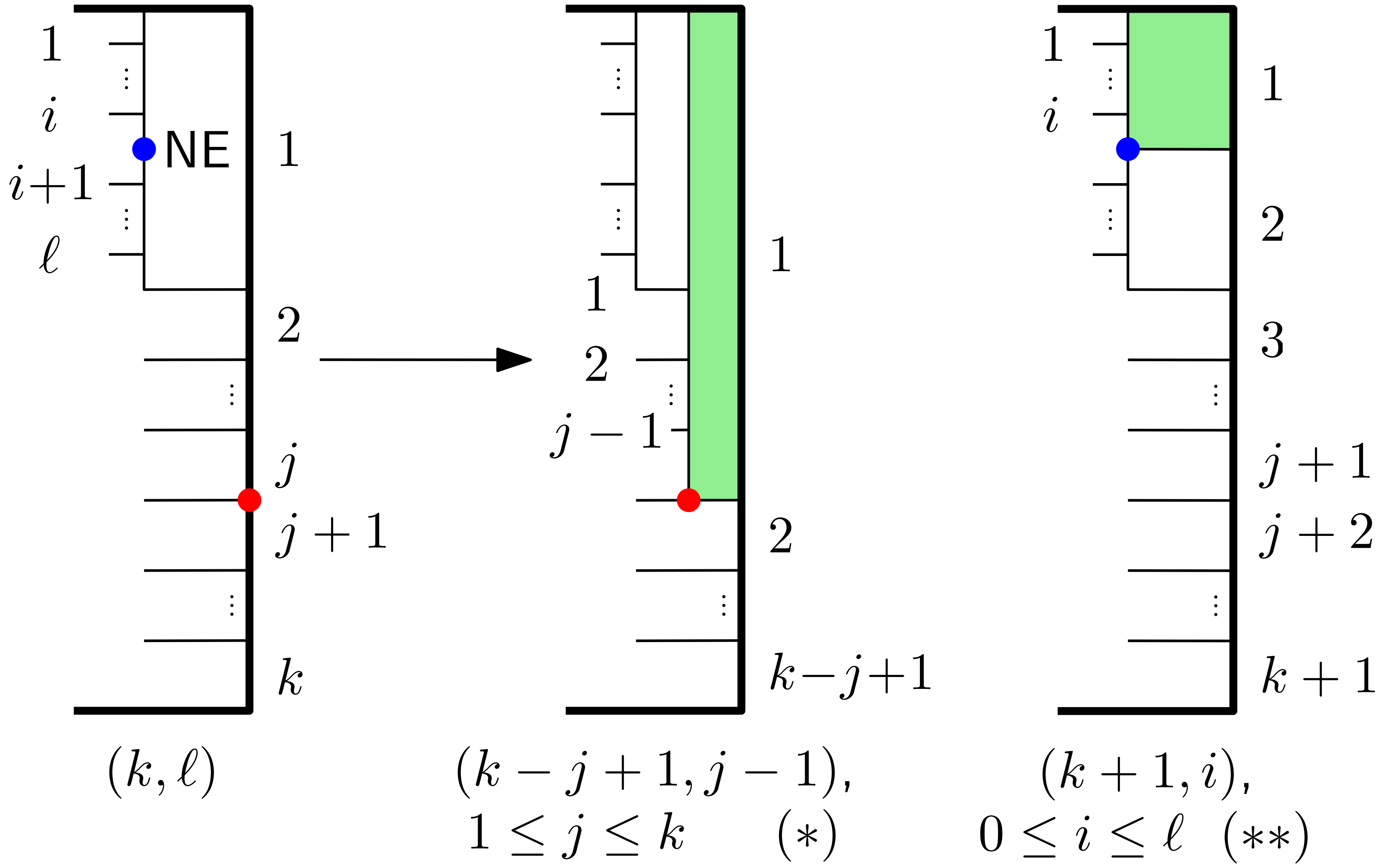
Succession rules : $(k,\ell) \longrightarrow (1, k-1),\ (2, k-2),\ \ldots,\ (k, 0);$ $(*)$

$(k+1, \ell),\ (k+1, \ell-1),\ \ldots,\ (k+1, 0).$ $(**)$



$k = n - M$ bounce

$\ell = \#$ admissible values j, $0 < j < e_n$.



$k = \#$ rectangles that touch E;

$\ell = \#$ segments that touch NE on the left

T1 is a generating tree for $I(010, 101, 120, 201)$ and for ⊤-avoiding rectangulations

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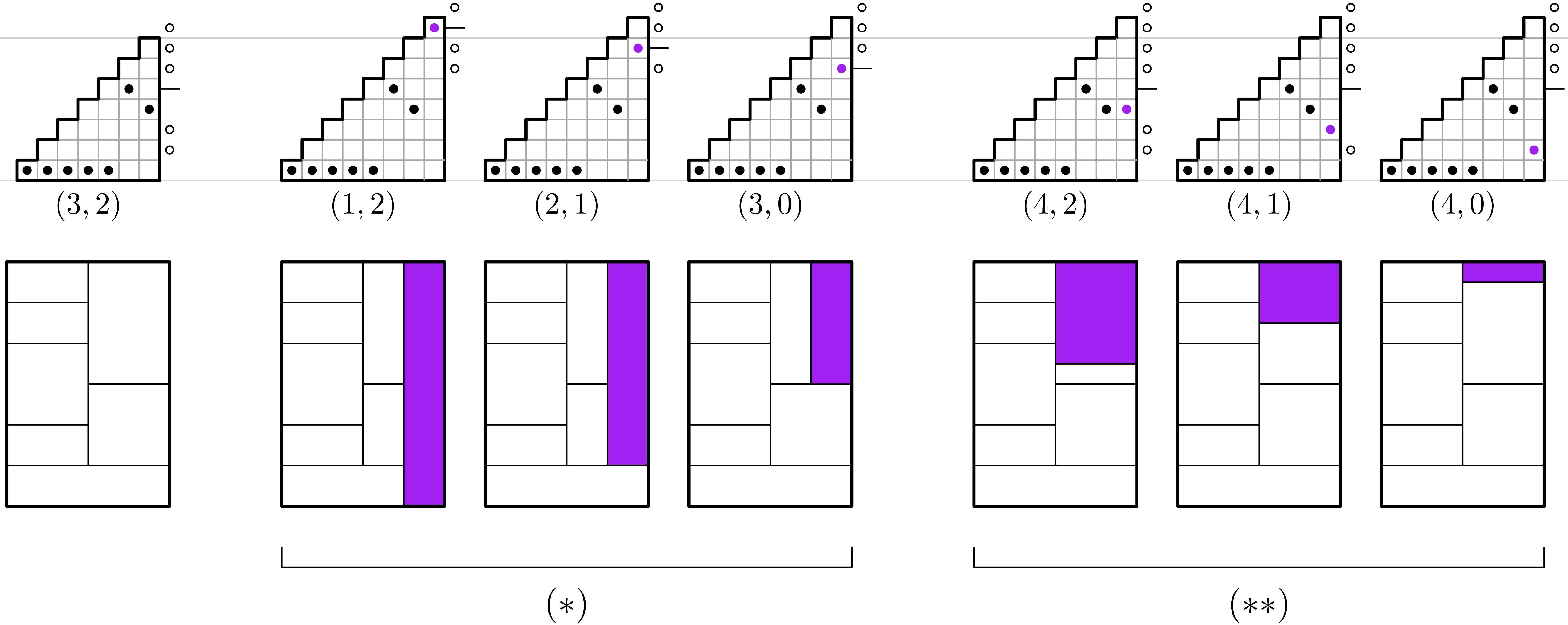

T2 is a generating tree for $I(011, 201)$ and for $\perp$-avoiding rectangulations

Root : $(1, 0)$.

Succession rules : $(k, \ell) \longrightarrow (1, k+\ell-1),\ (2, k+\ell-2),\ \ldots,\ (k, \ell);$ $(*)$

$(k+1, \ell-1),\ (k+1, \ell-2),\ \ldots,\ (k+1, 0);$ $(**)$

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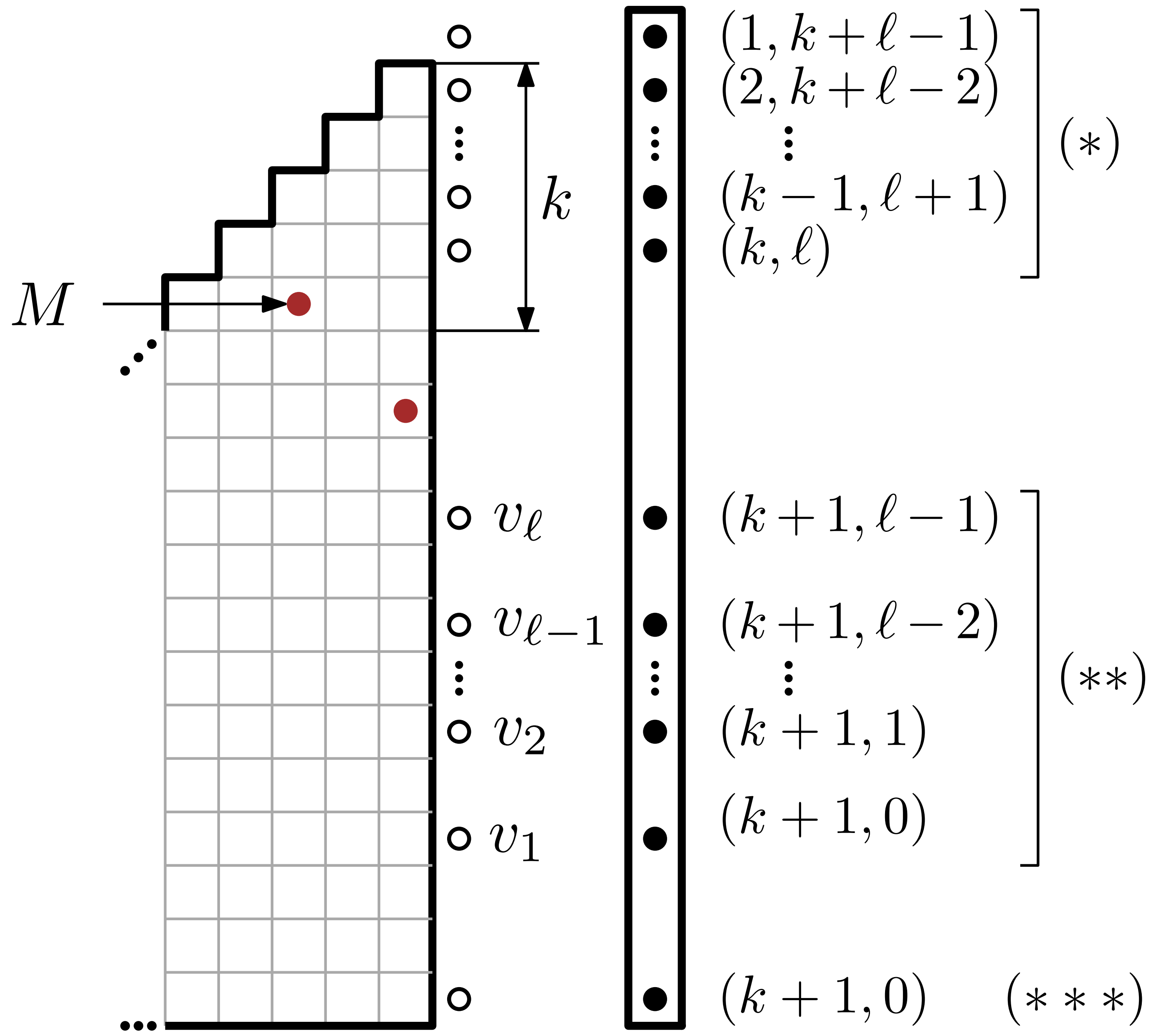

$k = n - M$ bounce

$\ell = \#$ admissible values j, $0 < j < M$.

T2 is a generating tree for $I(011, 201)$ and for ⊥-avoiding rectangulations

Root : $(1, 0)$.

Succession rules : $(k, \ell) \longrightarrow (1, k + \ell - 1),\ (2, k + \ell - 2),\ \ldots,\ (k, \ell);$ $(*)$

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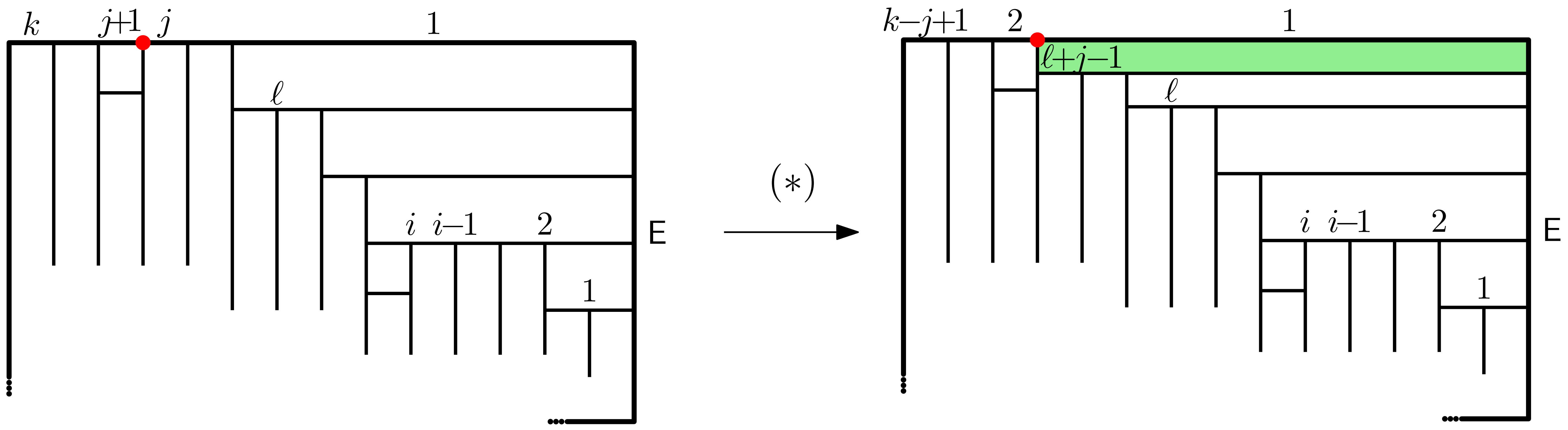

(k, ℓ) $(k - j + 1, \ell + j - 1),\quad 1 \leq j \leq k$

$k = \#$ rectangles that touch N;

$\ell = \#$ vertical segments whose upper endpoint lies on a horitontal segment that reaches E

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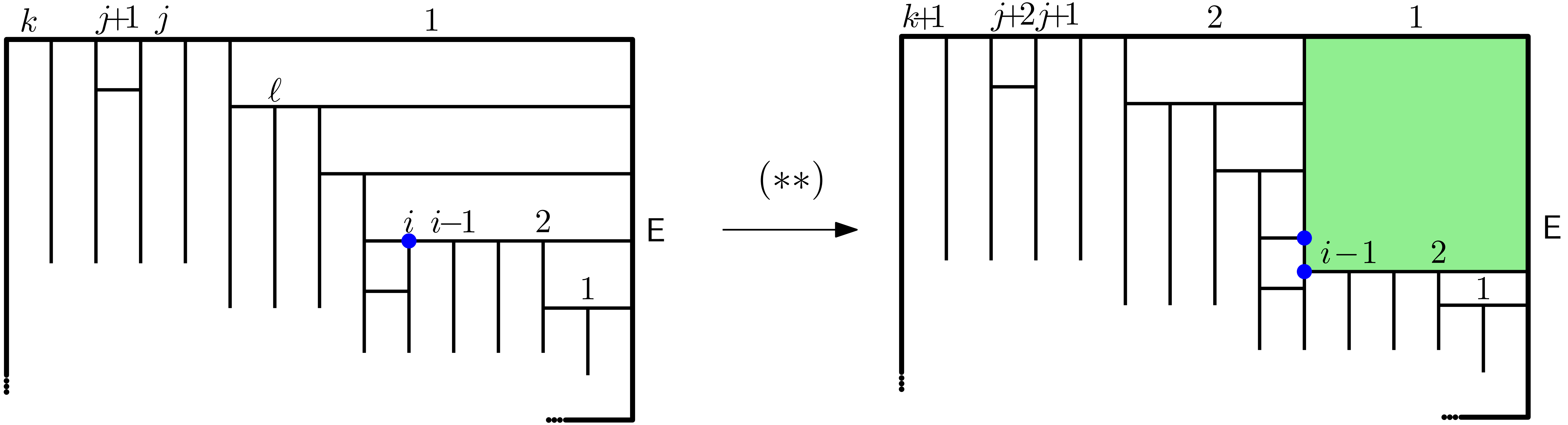

(k, ℓ) $\qquad\qquad$ $(k+1, i-1), \quad 1 \leq i \leq \ell$

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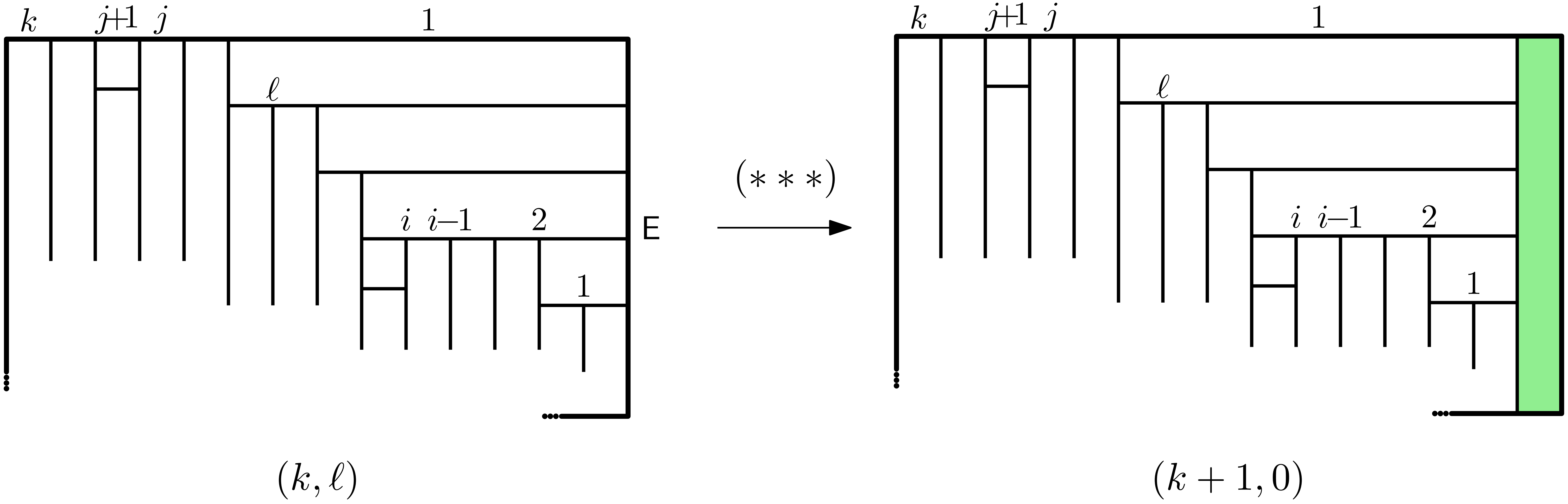

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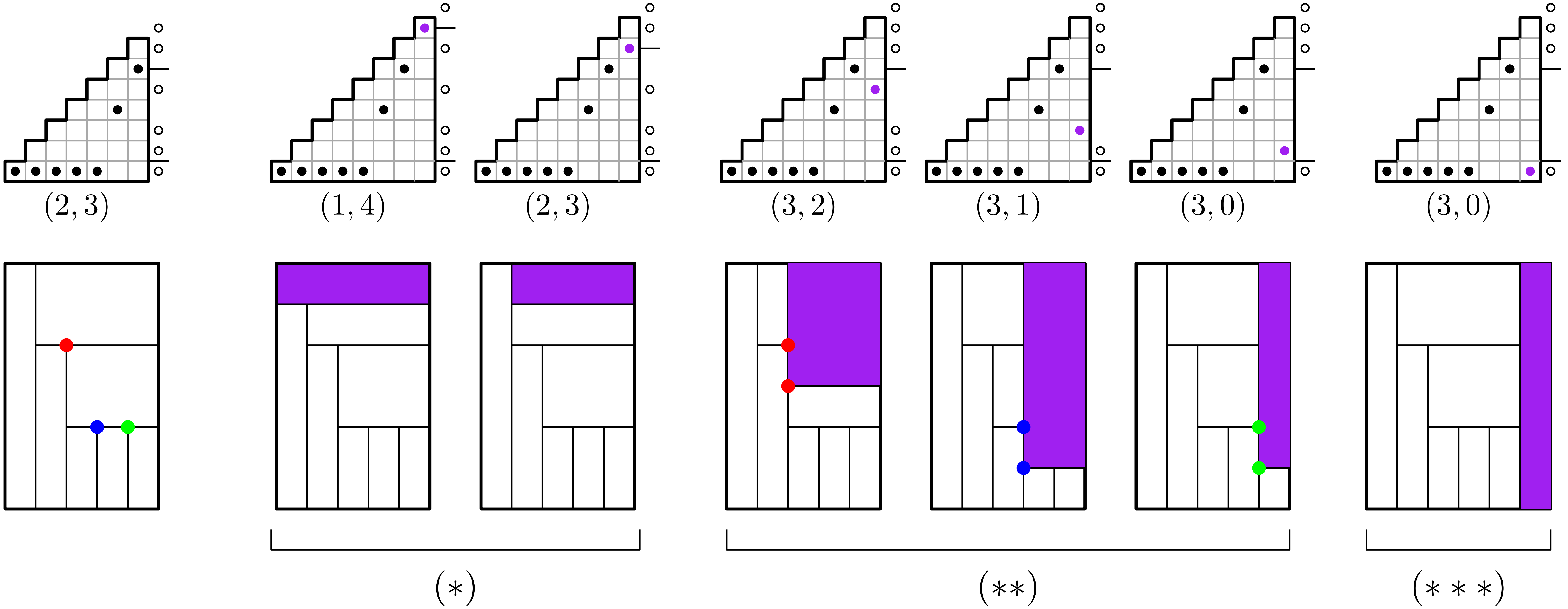
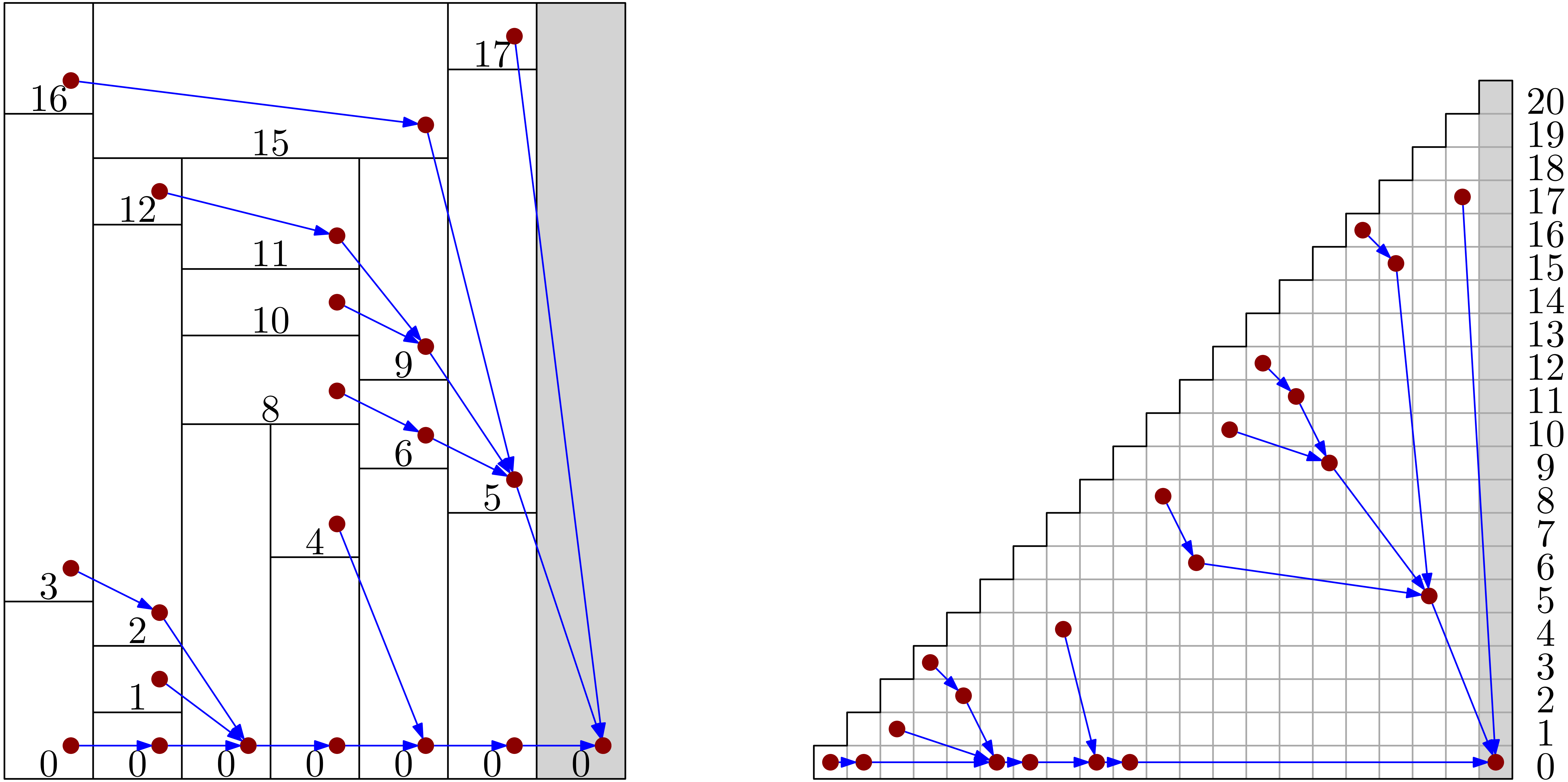


Explicit bijection between $I(011, 201)$ and $\perp$-avoiding rectangulations



The main result

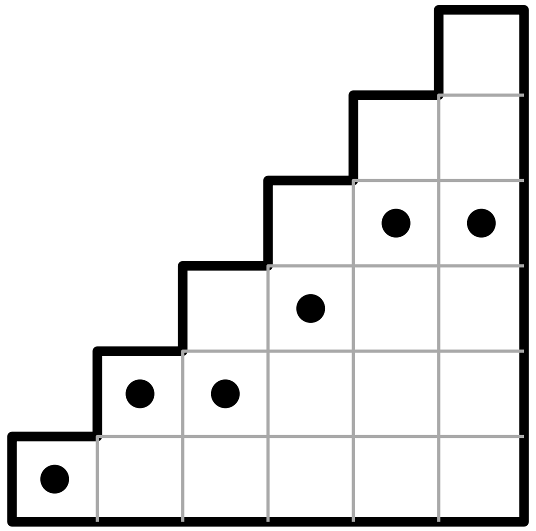
Theorem. For every $n \geq 1$:

1. We have $|I_n(010, 101, 120, 201)| = |I_n(011, 201)|$.
2. The quadruple of statistics (a, b, c, d) for $I_n(010, 101, 120, 201)$, $I_n(010, 110, 120, 210)$, and $I_n(010, 100, 120, 210)$, where

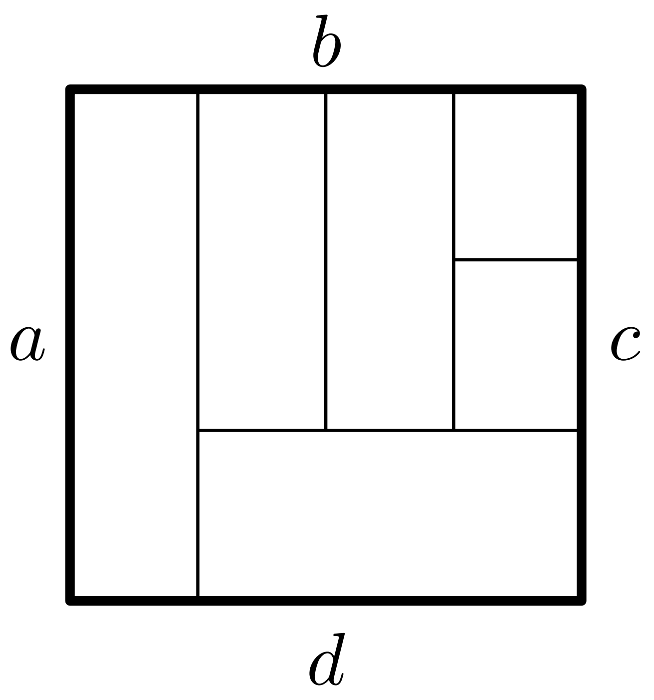
   a is the number of 0 elements, b is the number of left-to-right-maxima,
   c is the bounce, d is the number of high elements.

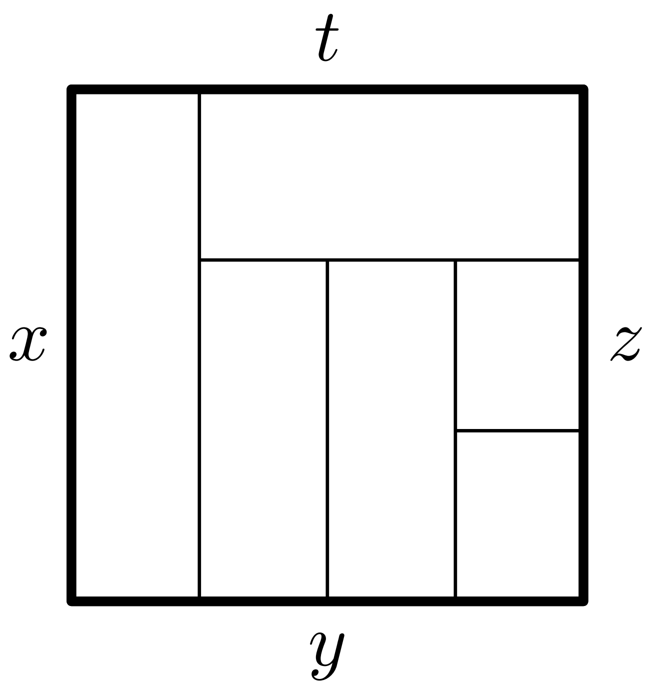
   matches the quadruple of statistics (x, y, z, t) for $I_n(011, 201)$, where

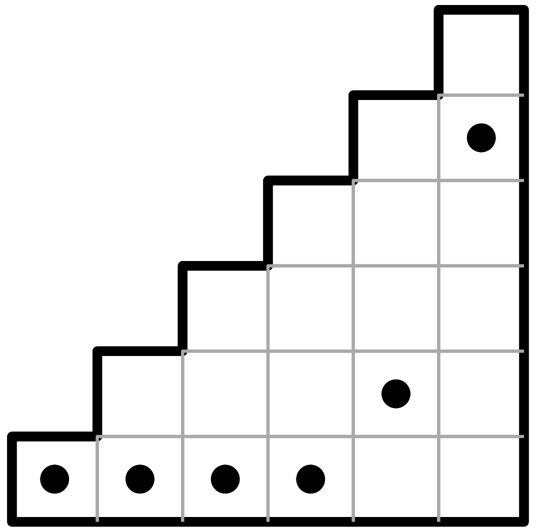
   x is the number of high elements, y is the number of 0 elements,
   z is the number of right-to-left-minima, t is the bounce.



$I_n(010, 101, 120, 201)$



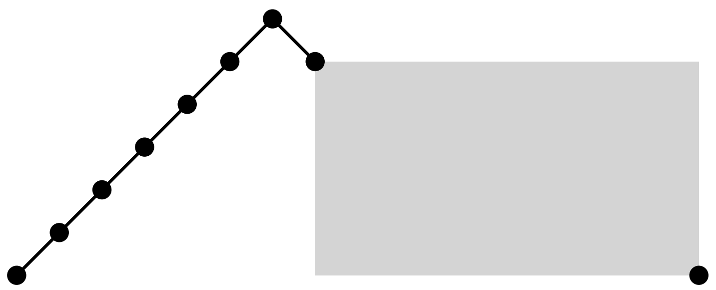


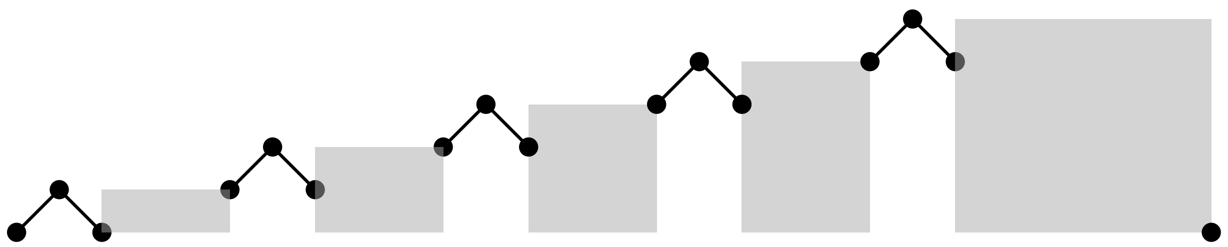


$I_n(011, 201)$

Rushed Dyck paths and progressive Dyck paths

A Dyck path is *rushed* if it starts with $k \geq 1$ up-steps and then never visits the altitude k again.
A Dyck path is *progressive* if every peak at height $h > 1$ is preceded by at least one peak at height $h-1$.





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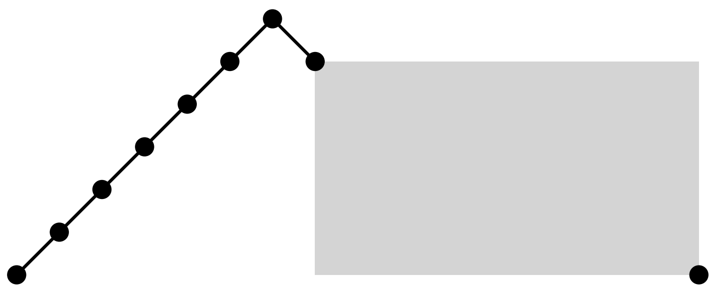

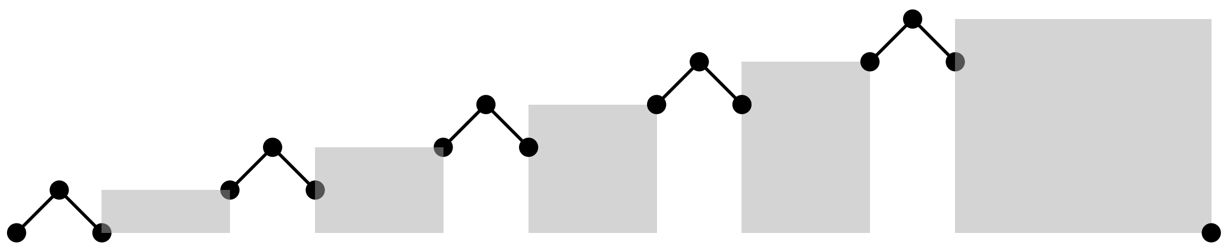

Both families are enumerated by A287709. (OEIS description: the second family. AA 2014: Two families are equinumerous, proof with generating functions. Jelínek 2016: Involution-based proof. Bacher 2024: A bijective proof, enumeration, asymptotics.)

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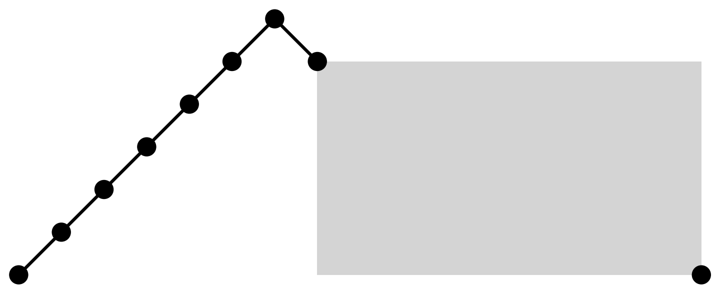

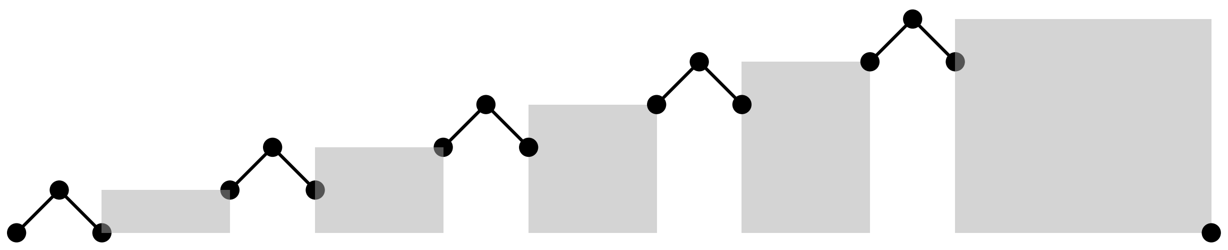

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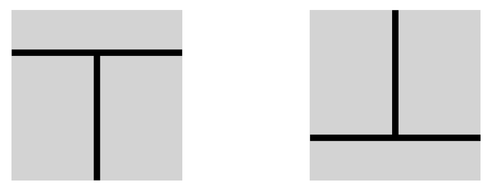

weak: 2^{n-1}
strong: OEIS A287709

Rushed Dyck paths and $(\vdash, \dashv)$-avoiding strong rectangulations

Theorem. For all $n, k \geq 1$, the number of rushed Dyck paths of semilength $n+1$ and height $k+1$ is equal to the number of $(\vdash, \dashv)$-avoiding strong rectangulations of size n with $k-1$ horizontal segments.

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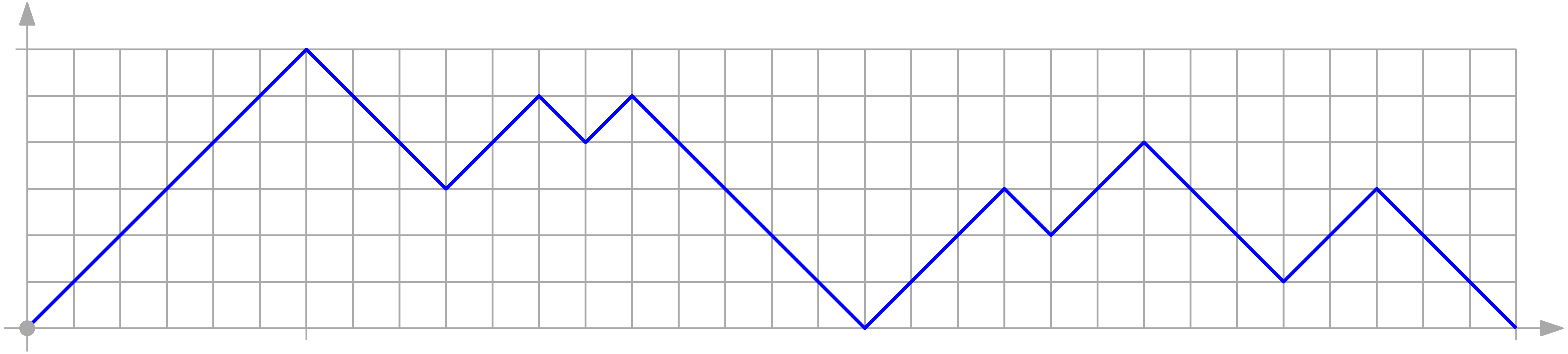

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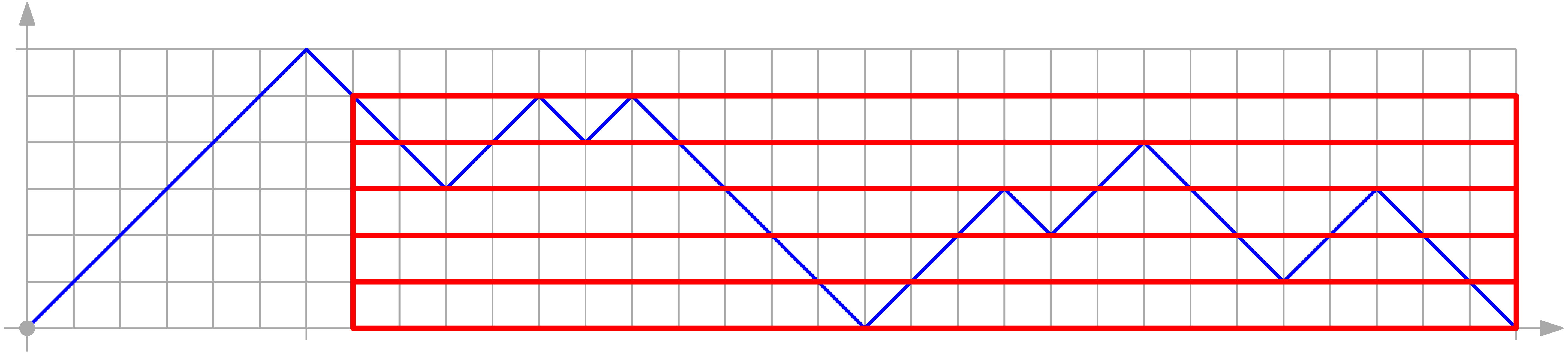

Rushed Dyck paths and $(\vdash, \dashv)$-avoiding strong rectangulations

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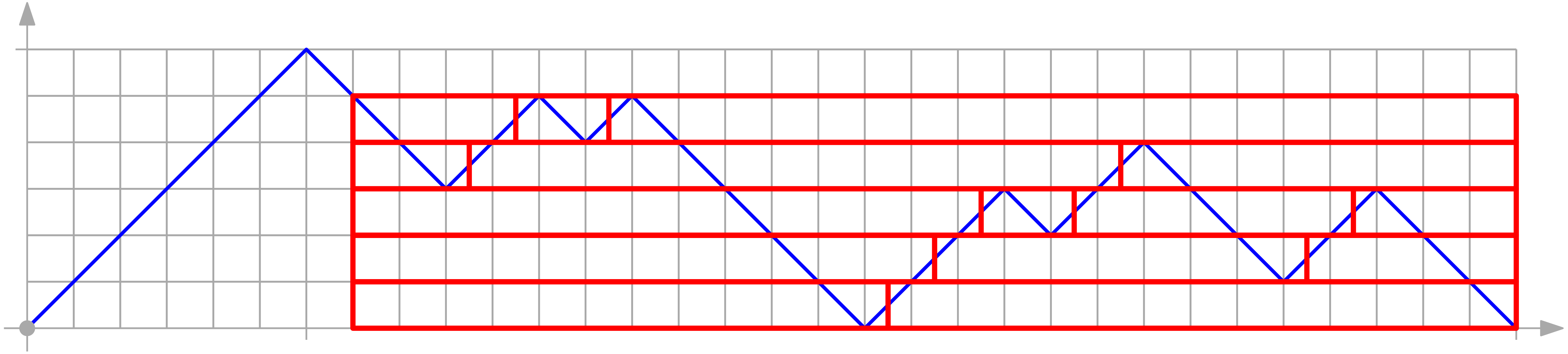

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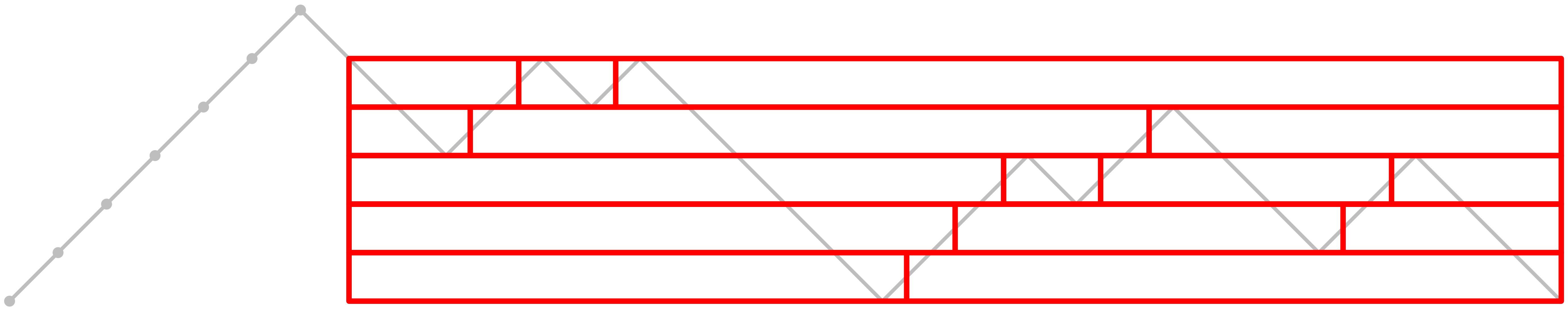

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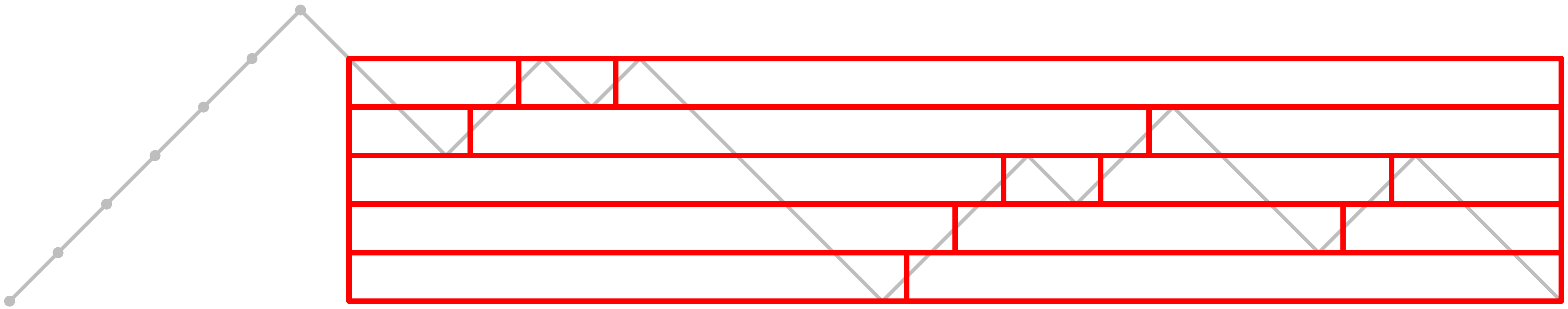

$(\vdash, \dashv)$-avoiding rectangulations are precisely those $(\vdash)$-avoiding rectangulations in which the number of rectangles that touch E is equal to the number of rectangles that touch W. Combining this observation with the main theorem, we obtain:

- $(010, 101, 120, 201)$-avoiding inversion sequences in which all left-to-right maxima are high, are enumerated by A287709.
- $(011, 201)$-avoiding inversion sequences in which the bounce k is equal to the number of 0-elements (or, equivalently: in which the set of values is precisely $\{0, 1, \ldots, M\}$) are enumerated by A287709.

Follow-ups:

- Patterns in rectangulations and permutation patterns (next talk).
- Catalogue of pattern-avoidance classes of rectangulations (next conference?).