Monotone Grid Classes

Limit Shapes and Enumeration

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Co-conspirators

Some of this talk is based on joint work with
Noura Alshammari.

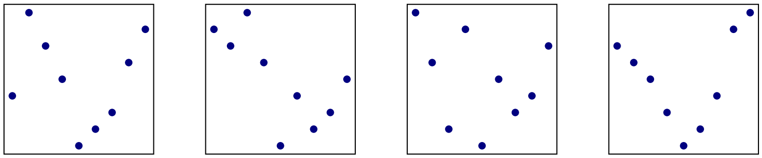
Another small part is based on joint work with
Robert Brignall and **Nik Ruškuc**.

Monotone grid classes

Monotone grid classes:

- Grid(M) is defined by a gridding matrix M.
- Entries of M are drawn from {⧄, ⧅, □}.
- Entries corresponds to cells in M-griddings of permutations.
  - ⧄ Any points in the cell must increase.
  - ⧅ Any points in the cell must decrease.
  - □ Blank cells must be empty.

Four permutations in Grid(⧅⧅⧄ / □⧄□)

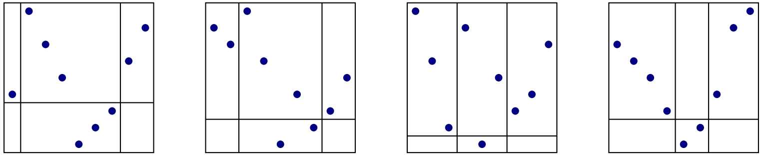


Monotone grid classes

Monotone grid classes:

- Grid(M) is defined by a gridding matrix M.
- Entries of M are drawn from $\{\nearrow, \searrow, \square\}$.
- Entries corresponds to cells in M-griddings of permutations.
  - $\nearrow$ Any points in the cell must increase.
  - $\searrow$ Any points in the cell must decrease.
  - $\square$ Blank cells must be empty.

Four permutations in Grid$\left(\begin{smallmatrix} \searrow & \searrow & \nearrow \\ & \nearrow & \end{smallmatrix}\right)$, each witnessed by a $\begin{smallmatrix} \searrow & \searrow & \nearrow \\ & \nearrow & \end{smallmatrix}$-gridding



Limit shapes and enumeration

Some questions

- ? Limit shape
  What does a typical large permutation in a given monotone grid class look like?

- ? Enumeration (exact and asymptotic)
  What is known?
  ★ **Open questions**: What is not known?

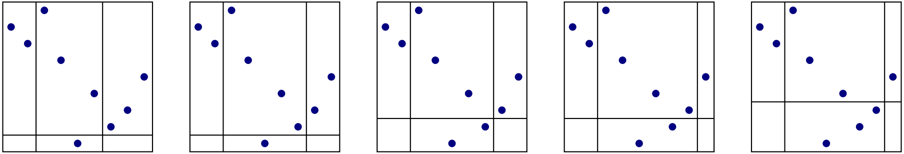
- ? How are these questions related?

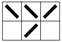
Multiple griddings

A permutation in Grid(M) may have more than one M-gridding.

- Row/columns dividers may be positioned in more than one way.
- **This** is what makes the analysis of grid classes hard.

The five  -griddings of 879614235



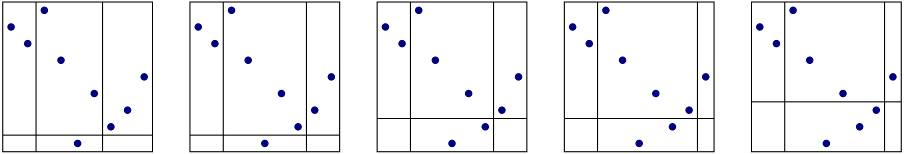
- These correspond to **one** element of Grid().

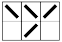
Classes of gridded permutations

It's much easier to analyse gridded permutations.

- Grid$^{\#}(M)$ is the gridded class, consisting of M-gridded permutations.

The five members of Grid$^{\#}$() with underlying permutation 879614235



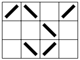
- These are **five** distinct elements of Grid$^{\#}$().

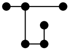
Cell graphs and connected classes

Cell graph of Grid(M):

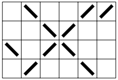
- Vertices: non-blank cells of M.
- Edges join vertices sharing a row or column.
- Properties of the cell graph are transferred to the class.
- Mostly, we focus on connected classes.

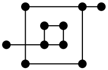
Connected acyclic class





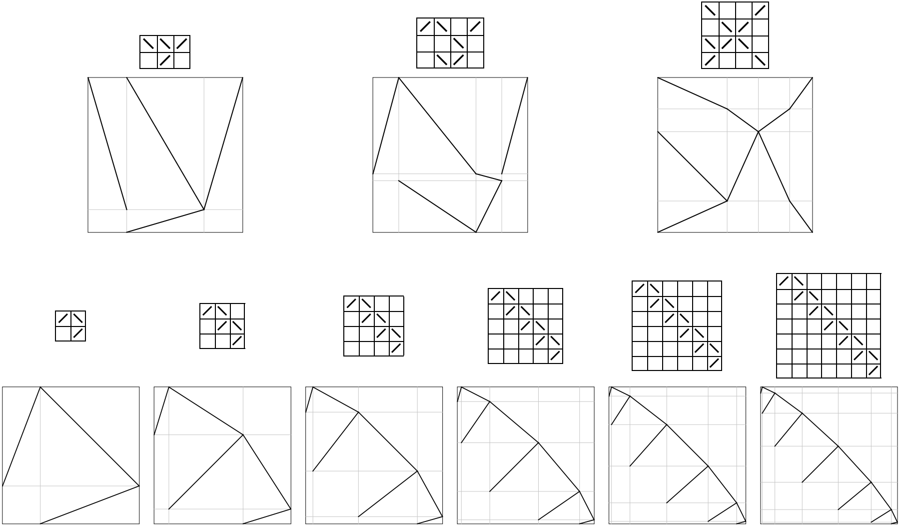
Two-component pseudoforest† class





†At most one cycle per component

Part I: Limit Shapes



Based on B. (2015), Albert & Vatter (2019), and Alshammari & B. (2025).

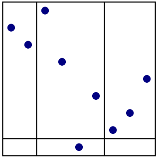
Counting points in cells

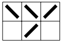
M-admissible matrix

- Nonnegative matrix $A = (a_{i,j})$ such that $a_{i,j} = 0$ if $M_{i,j}$ is blank.

Integer M-admissible matrix: records **number** of points in each cell.

- If $A = (a_{i,j})$, then $\mathrm{Grid}^{\#}_{A}(M)$ consists of M-gridded permutations with $a_{i,j}$ points in cell (i, j).



$\in \quad \mathrm{Grid}^{\#}_{\begin{pmatrix}2 & 3 & 3\\ 0 & 1 & 0\end{pmatrix}}\Big($  $\Big)$

Counting points in cells

Enumerating $\mathsf{Grid}^{\#}_{A}(M)$ is easy.

Proposition

If M has dimensions $r \times s$, then

$$\left|\mathsf{Grid}^{\#}_{A}(M)\right| = \prod_{i=1}^{r} \binom{\sum_{j=1}^{s} a_{i,j}}{a_{i,1}, a_{i,2}, \ldots, a_{i,s}} \times \prod_{j=1}^{s} \binom{\sum_{i=1}^{r} a_{i,j}}{a_{1,j}, a_{2,j}, \ldots, a_{r,j}}.$$

- Each multinomial coefficient counts the possibilities for one row or column.
- The ordering of points (increasing or decreasing) *within* a particular cell is fixed by the corresponding entry of M.
- The interleaving of points in *distinct* cells in the same row or column can be chosen arbitrarily and independently.

The distribution of points between cells

M-distribution matrix

- M-admissible matrix whose entries sum to one.

M-distribution matrix: records **proportion** of points in each cell.

- If $\Gamma = (\gamma_{i,j})$ is an M-distribution matrix, then

  $$\mathsf{Grid}^{\#}_{\Gamma n}(M) = \mathsf{Grid}^{\#}_{A}(M) \text{ for some } A = (a_{i,j})$$

  such that $\sum a_{i,j} = n$ and $|a_{i,j} - n\gamma_{i,j}| < 1$.

  - The existence of A is guaranteed by Baranyai's Rounding Lemma.
  - If $\sigma^{\#} \in \mathsf{Grid}^{\#}_{\Gamma n}(M)$ the proportion of points of $\sigma^{\#}$ in cell (i,j) differs from $\gamma_{i,j}$ by less than $1/n$.

The distribution of points between cells

By Stirling's approximation:

Proposition

If $\Gamma = (\gamma_{i,j})$ is an M-distribution matrix with row sums $\rho_i = \sum_j \gamma_{i,j}$ and column sums $\kappa_j = \sum_i \gamma_{i,j}$, then

$$\left|\mathsf{Grid}^{\#}_{\Gamma n}(M)\right| \;\sim\; C n^{\beta} g^{n},$$

where

$$g \;=\; g(\Gamma) \;:=\; \prod_i \frac{\rho_i^{\rho_i}}{\prod_j \gamma_{i,j}^{\gamma_{i,j}}} \times \prod_j \frac{\kappa_j^{\kappa_j}}{\prod_i \gamma_{i,j}^{\gamma_{i,j}}},$$

and C and β are constants that only depend on Γ.

Maximising the growth rate

Given Grid(M), we would like to find a maximal M-distribution matrix Γ for which the growth rate $g(\Gamma)$ is greatest.

Proposition

Suppose $\Gamma = (\gamma_{i,j})$ is a maximal M-distribution matrix. Then there exists a constant λ such that, for each nonzero entry $\gamma_{i,j}$ of Γ, we have

$$\frac{\gamma_{i,j}^2}{\rho_i \kappa_j} = \lambda,$$

where $\rho_i = \sum_j \gamma_{i,j}$ and $\kappa_j = \sum_i \gamma_{i,j}$ are the row and column sums of Γ.

- Proof uses Lagrange multipliers to solve the constrained optimisation problem.

Unique maximal distribution for connected classes

For **connected** classes, there is only one maximal distribution.

Proposition

If Grid(M) *is a connected grid class, then it has a unique maximal M-distribution matrix* Γ_M.

- Proof uses linear algebra (singular value decomposition and Perron–Frobenius).

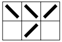
The set of equations

$$\sum_{i,j} \gamma_{i,j} = 1, \qquad \frac{\gamma^2_{i_1,j_1}}{\rho_{i_1} \kappa_{j_1}} = \frac{\gamma^2_{i_2,j_2}}{\rho_{i_2} \kappa_{j_2}} = \frac{\gamma^2_{i_3,j_3}}{\rho_{i_3} \kappa_{j_3}} = \ldots = \lambda$$

has a unique positive solution.

Unique maximal distribution for connected classes

Example

$M =$ . $\Gamma_M = \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \delta & 0 \end{pmatrix}$.

Maximal M-distribution matrix satisfies $\alpha + \beta + \gamma + \delta = 1$ and

$$\frac{\alpha^2}{\alpha(\alpha+\beta+\gamma)} = \frac{\beta^2}{(\beta+\delta)(\alpha+\beta+\gamma)} = \frac{\gamma^2}{\gamma(\alpha+\beta+\gamma)} = \frac{\delta^2}{\delta(\beta+\delta)}.$$

Solution:

$$\Gamma_M = \begin{pmatrix} \frac{1}{4} & \frac{1}{2\sqrt{2}} & \frac{1}{4} \\ 0 & \frac{1}{4}(2-\sqrt{2}) & 0 \end{pmatrix}.$$

The typical distribution of points

For connected classes, the distribution of points in almost all gridded permutations in $\mathsf{Grid}^{\#}(M)$ is close to Γ_M.

- $\sigma^{\#}_{(i,j)}$: the number of points of $\sigma^{\#}$ in cell (i,j).

Theorem

If $\mathsf{Grid}(M)$ *is connected and* $\Gamma_M = (\gamma_{i,j})$*, then for any* $\varepsilon > 0$*,*

$$\lim_{n\to\infty} \mathbb{P}\Big[\max_{i,j} \big|\boldsymbol{\sigma}^{\#}_{(i,j)}/n - \gamma_{i,j}\big| \leqslant \varepsilon\Big] \;=\; 1,$$

where, for each n*, we draw* $\boldsymbol{\sigma}^{\#}$ *uniformly at random from* $\mathsf{Grid}^{\#}_n(M)$*.*

- Proved by showing that the set of gridded permutations with distributions ε-far from Γ_M have a smaller growth rate.

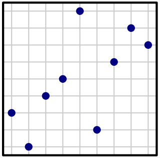
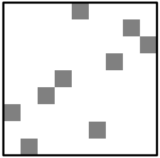
Permutons

Definition (permuton)

Probability measure μ on the unit square $[0,1]^2$ with uniform marginals:

$$\mu([a,b] \times [0,1]) = \mu([0,1] \times [a,b]) = b - a \text{ for every } 0 \leqslant a \leqslant b \leqslant 1.$$

Permuton μ_π corresponding to permutation π

$\pi =$  $\mu_\pi =$ 

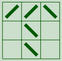
- Mass for each point: $1/n$
- Small square area: $1/n^2$
- Density ("height"): n

Limit shapes

Definition (limit shape of a permutation class)

The permuton μ is the limit shape of $\mathcal{C}$ if the sequence of random permutons $(\mu_{\boldsymbol{\sigma}_n})_{n\geqslant 1}$ converges in distribution for the weak topology to μ, where, for each n, we draw $\boldsymbol{\sigma}_n$ uniformly at random from $\mathcal{C}_n$.

- Formalises what a typical large permutation looks like.

Example ()

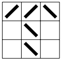
60 120 180 240 limit shape

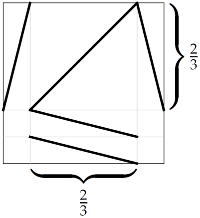
Limit shapes

Two final steps:

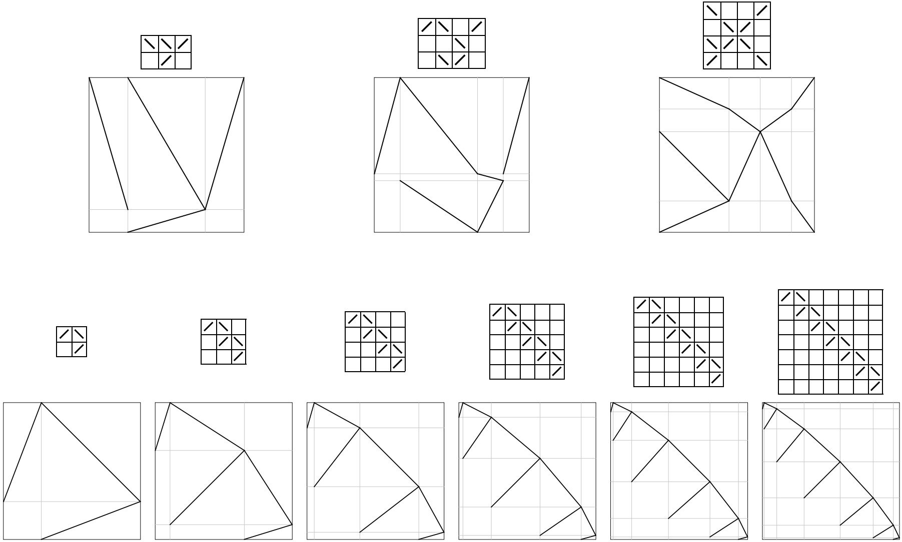
- In almost all large gridded permutations, the points are close to the diagonals across the cells.
- The limit shape of $\mathsf{Grid}(M)$ is the same as that for $\mathsf{Grid}^{\#}(M)$.

Example

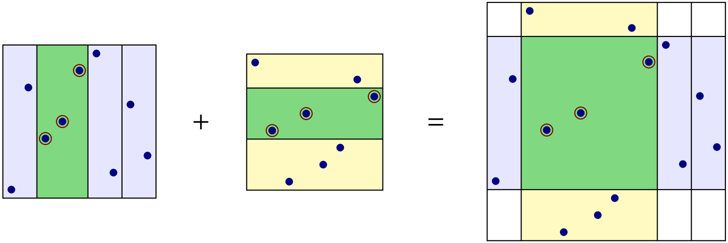
$M =$  $\implies \quad \Gamma_M = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & 0 \end{pmatrix}.$

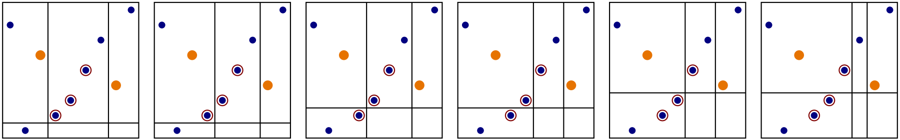


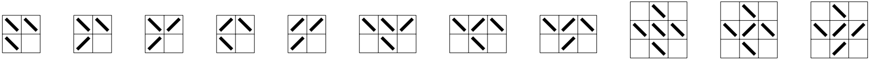
Limit shapes



Part II: Enumeration







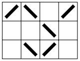
L_0 L_1 L_3 L_4 L_7 T_2 T_4 T_5 X_0 X_4 X_7

Growth rates

The exponential growth rate is known for any grid class.

Theorem (B.; Albert & Vatter)

Let B be the binary matrix with $B_{i,j} = 0$ if $M_{i,j} = \square$ and $B_{i,j} = 1$ otherwise. Then, $\operatorname{gr}(\mathsf{Grid}(M))$ exists and is equal to the largest eigenvalue of $B^{\mathsf{T}}B$.

$M =$  $\implies B = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$

- If M is connected, then $\operatorname{gr}(\mathsf{Grid}(M)) = g(\Gamma_M)$, the growth rate of gridded permutations with maximal distribution.

Bases (a brief digression)

Theorem (Albert, Atkinson, Bouvel, Ruškuc & Vatter)

*Every **acyclic** monotone grid class is finitely based.*

- Because acyclic classes are geometric classes.

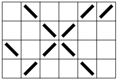
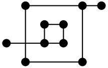
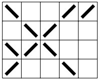
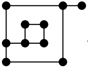
Theorem (B., Brignall & Ruškuc)

*Every **unicyclic**† monotone grid class is finitely based.*

†Exactly one cycle

Theorem (B., Brignall & Ruškuc)

*Some monotone grid classes **with two cycles** are not finitely based, such as*

  *and*   .

Types of generating function

Theorem (Albert, Atkinson, Bouvel, Ruškuc & Vatter)

*Every **acyclic** monotone grid class has a **rational** generating function.*

- Because acyclic classes are geometric classes.

Theorem (Gridded classes, B.)

*Every **pseudoforest gridded** class has an explicitly constructible **algebraic** generating function.*

*Every **gridded** class has a **D-finite** generating function.*

★ Conjectures (Grid classes, B.)

Every pseudoforest monotone grid class has an algebraic generating function.

Every monotone grid class has a D-finite generating function.

Exact enumeration

Polynomial (Homberger & Vatter)

- At most one ⧄ or ⧅ in any column or row.
- Algorithm to give polynomial enumeration (growth rate equals 1).

Skinny (B.; Brignall & Sliačan)

- Procedure to give g.f. when M has dimensions $1 \times k$.
- Special case (Asinowski, Banderier & Hackl):

$$\mathsf{Grid}(\underbrace{⧄⧄\cdots⧄}_{k}) : \quad \sum_{r=1}^{k} \frac{1}{1-rz}\left(\frac{rz}{rz-1}\right)^{k-r}.$$

Small acyclic (Hušek & Opler, after Braunfeld)

- Using monadic second-order logic.

Exact enumeration

★ 2×2 classes

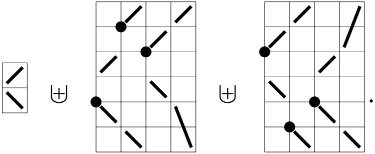
- Growth rate: 4.
- Asymptotics have the form $c4^n/\sqrt{n}$.

		g.f.	**basis**
C_0		?	conjecture (35 perms)
C_1		?	conjecture (36 perms)
C_2		?	conjecture (10 perms)
C_3		conjecture (A163824)	conjecture (14 perms)
C_6		Atkinson A029759	$\text{Av}(2143, 3412)$
C_9		?	conjecture (32 perms)

Double chevron class

★ Conjecture (“entanglement diagrams”)

Every permutation in Grid() has a **unique** gridding in exactly one of the following three diagrams:



- There must be a point at each •.

Monotone curve classes

★ **A new family of permutation classes** (superset of grid classes) ★

Definition (monotone curve class)

Suppose $\mathcal{S} = \{\{p_1, q_1\}, \{p_2, q_2\}, \ldots, \{p_k, q_k\}\}$ is a finite multiset of pairs of lattice points $\{p_i, q_i\} \in \mathbb{N}^2$ sharing neither x or y coordinate.

Then $\mathsf{Mono}(\mathcal{S})$ consists of those permutations that can be drawn on k monotone curves joining each p_i to q_i.

- $\mathsf{Av}(321) = \mathsf{Mono}\big(\{(0,0),(1,1)\}, \{(0,0),(1,1)\}\big)$.
  - Two increasing sequences.
- What is the basis and enumeration of Mono()?
- The relationship between grid and curve classes is similar to that between geometric and picture classes.

Asymptotic enumeration

Connected acyclic and unicyclic (Alshammari & B.)

- Procedure to give asymptotics $\mathsf{Grid}(M) \sim g^n \theta(n)$, where g is the exponential growth rate, and $\theta(n)$ is subexponential.

Recipe

1. Typical distribution of points (Γ_M, as above).
2. Gridded asymptotics $\left|\mathsf{Grid}_n^{\#}(M)\right| \sim g^n \theta^{\#}(n)$.
3. Structure of typical $\sigma^{\#}$ if σ has exactly ℓ distinct M-griddings.
4. With $\boldsymbol{\sigma}_n^{\#}$ drawn uniformly from $\mathsf{Grid}^{\#}(M)$, let

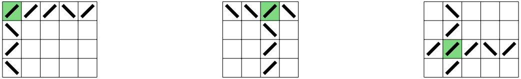
$$P_\ell \;=\; \lim_{n\to\infty} \mathbb{P}\big[\boldsymbol{\sigma}_n \text{ has exactly } \ell \text{ distinct } M\text{-griddings}\big].$$

5. Then, $\left|\mathsf{Grid}_n(M)\right| \sim \kappa g^n \theta^{\#}(n)$, where $\kappa = \sum_{\ell \geqslant 1} P_\ell/\ell$.

Connected one-corner classes

A cell is a corner if it isn't the only non-blank cell in its row or column.

Connected one-corner classes: L-shaped, T-shaped or cross-shaped:



- We assume $r + 1$ rows and $c + 1$ columns.

1. Asymptotic distribution in one-corner classes

Grid(M) connected with one-corner and dimensions $(r+1)\times(c+1)$:

$$\Gamma_M = \begin{pmatrix} 0 & \gamma & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 & 0 \\ \beta & \alpha & \beta & \beta & \beta \\ 0 & \gamma & 0 & 0 & 0 \end{pmatrix},$$

satisfying

$$\alpha + c\beta + r\gamma = 1 \quad \text{and} \quad \frac{\alpha^2}{(\alpha+c\beta)(\alpha+r\gamma)} = \frac{\beta}{\alpha+c\beta} = \frac{\gamma}{\alpha+r\gamma} = \lambda.$$

Solution

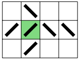
$$\alpha = \frac{1}{q}, \quad \beta = \frac{c-r+q-1}{2cq}, \quad \gamma = \frac{r-c+q-1}{2rq}, \quad \lambda = \frac{c+r+1-q}{2cr},$$

where

$$q = \sqrt{(c+r+1)^2 - 4cr}.$$

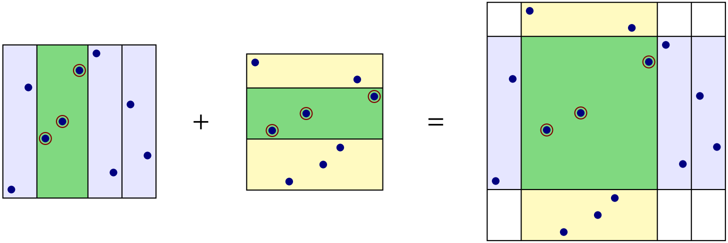
2. Enumerating acyclic and unicyclic gridded classes

We stitch together skinny classes at the corners.

Each gridded permutation in $\mathsf{Grid}^{\#}$ () is uniquely defined by

- a -gridded permutation and
- a -gridded permutation

with the same number of points in the corner.



Enumerating acyclic and unicyclic gridded classes

Stitching corresponds to an operation on generating functions.

- $\dfrac{1}{1-xz-cz}$ Horizontal $c+1$ cells; x counts corner points.
- $\dfrac{1}{1-yz-rz}$ Vertical $r+1$ cells; y counts corner points.

Stitching yields:

$$[z^0]\,\frac{1}{(1-x\sqrt{z}-cz)(1-\sqrt{z}/x-rz)} \;=\; \frac{1}{1-(c+r+1)z+crz^2}.$$

- By diagonalization of rational Laurent series (Stanley *Volume* 2).
  - Sum of residues at small poles.
- Repeat to give a **rational** g.f. for any **acyclic** class or an **algebraic** g.f. for any **unicyclic** class.

Enumerating gridded classes

Connected one-corner classes

If M is connected with one corner and dimensions $(r+1) \times (c+1)$, then

$$\sum_{n \geqslant 0} \left|\mathsf{Grid}_n^{\#}(M)\right| z^n = \frac{1}{1-(c+r+1)z+crz^2}.$$

Asymptotics

Hence (by standard analytic combinatorics),

$$\left|\mathsf{Grid}_n^{\#}(M)\right| \sim \theta^{\#} g^n,$$

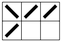
where

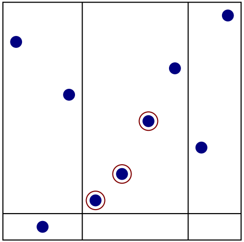
$$\theta^{\#} = \frac{c+r+q+1}{2q} \quad \text{and} \quad g = \frac{c+r+q+1}{2}.$$

3. Dancing

A point Q of an M-gridded permutation $\sigma^{\#}$ can dance if there is a sequence of one-step moves of row and column dividers, such that

- after each step the result is a valid M-gridding of σ, and
- at the end of the sequence some divider is on the other side of Q.

The 3 circled points in these -gridded permutations can dance:



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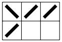
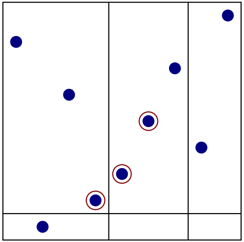
The 3 circled points in these -gridded permutations can dance:



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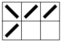
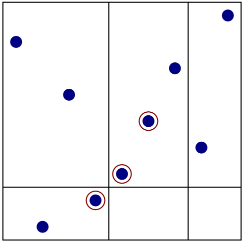
The 3 circled points in these -gridded permutations can dance:



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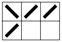
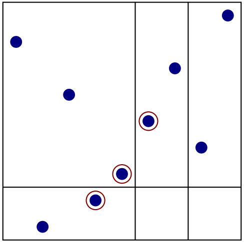
The 3 circled points in these -gridded permutations can dance:



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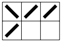
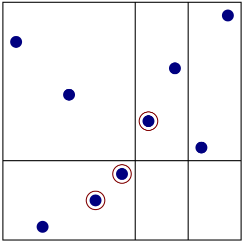
The 3 circled points in these -gridded permutations can dance:



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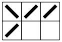
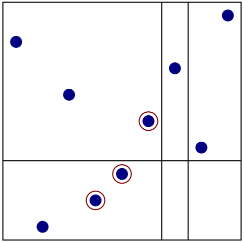
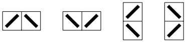
The 3 circled points in these -gridded permutations can dance:

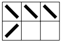


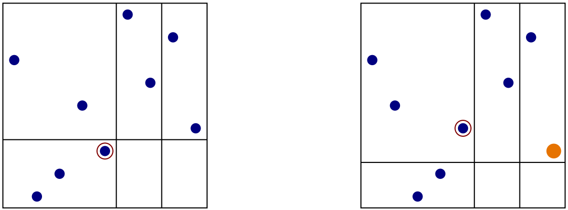
Peak dancing

Peak

Two adjacent cells with opposite orientation: 

A peak point (extremal point in a peak) can dance if it is adjacent to the divider, giving two griddings:

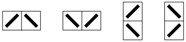
-gridded permutations, with peak points circled:



At the right, the orange controller prevents dancing.

Peak dancing

Peak

Two adjacent cells with opposite orientation: 

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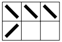
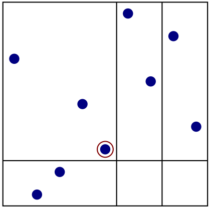
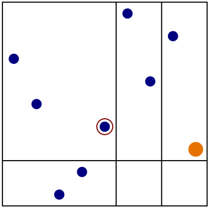
-gridded permutations, with peak points circled:

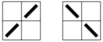




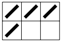
At the right, the orange controller prevents dancing.

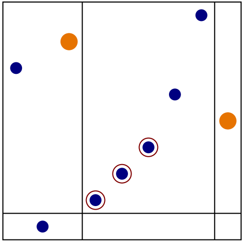
Diagonal dancing

Diagonal

Diagonally adjacent cells with the same orientation: 

An adjacent non-blank cell must have the same orientation:

The 3 circled points in these -gridded permutations can dance, giving four griddings:



Diagonal dancing

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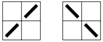
Diagonally adjacent cells with the same orientation: 

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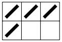
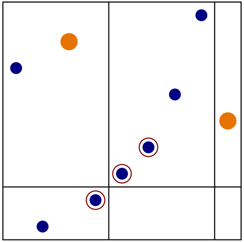
The 3 circled points in these -gridded permutations can dance, giving four griddings:



Diagonal dancing

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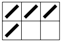
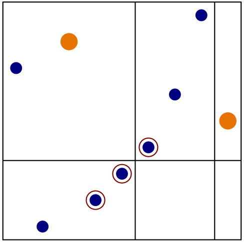
The 3 circled points in these -gridded permutations can dance, giving four griddings:



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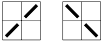
Diagonally adjacent cells with the same orientation: 

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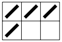
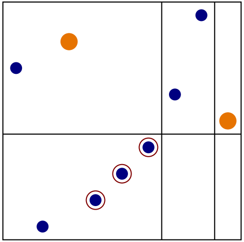
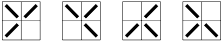
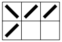
The 3 circled points in these -gridded permutations can dance, giving four griddings:

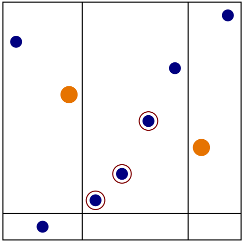


Tee dancing

Tee

Three adjacent cells forming a T shape: 

The 3 circled points in these -gridded permutations can dance, giving 6 griddings:



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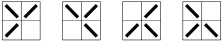
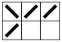
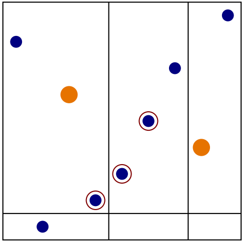
Three adjacent cells forming a T shape: 

The 3 circled points in these -gridded permutations can dance, giving 6 griddings:



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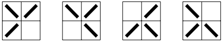
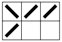
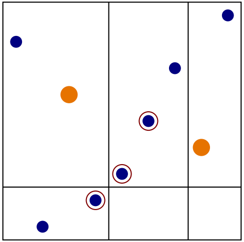
Three adjacent cells forming a T shape: 

The 3 circled points in these -gridded permutations can dance, giving 6 griddings:



Tee dancing

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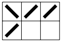
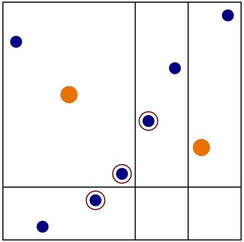
The 3 circled points in these -gridded permutations can dance, giving 6 griddings:



Tee dancing

Tee

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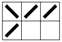
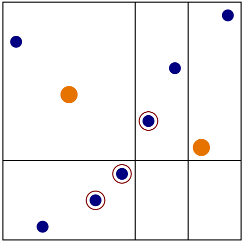
The 3 circled points in these -gridded permutations can dance, giving 6 griddings:



Tee dancing

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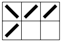
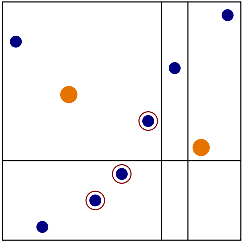
The 3 circled points in these -gridded permutations can dance, giving 6 griddings:



- Depending on the controllers,
  k dancers may give $2k-1$ or $2k$ or $2k+1$ griddings.

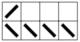
Constrained gridded permutations

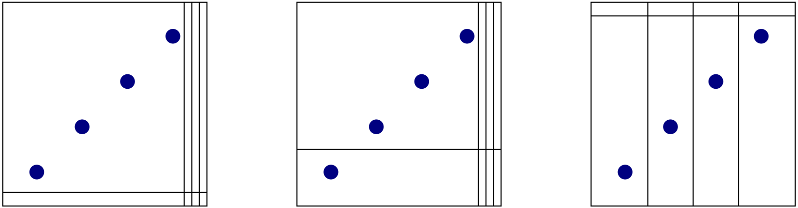
Definition (constrained)

An M-gridded permutation $\sigma^{\#}$ is **constrained** if every M-gridding of σ is the result of zero or more points of $\sigma^{\#}$ dancing.

In general, some M-griddings may not result from dancing.

Example

The three (unconstrained) -griddings of 1234:



Constrained gridded permutations

For connected M, almost all M-gridded permutations are constrained:

Theorem

If M is connected and $\boldsymbol{\sigma}_n^{\#}$ is drawn uniformly from $\mathsf{Grid}_n^{\#}(M)$, then

$$\lim_{n\to\infty} \mathbb{P}\big[\boldsymbol{\sigma}_n^{\#} \textit{ is } M\textit{-constrained}\big] \;=\; 1.$$

- Only unusually structured large permutations have griddings that aren't a result of dancing.

4 & 5. Counting griddings

If M is connected, for almost all M-gridded permutations:

- The distribution of points is close to Γ_M.
- The gridded permutation is constrained.

With $\boldsymbol{\sigma}_n^\#$ drawn uniformly from $\mathsf{Grid}^\#(M)$, for each $\ell \geqslant 1$, let

$$P_\ell \;=\; \lim_{n\to\infty} \mathbb{P}\big[\boldsymbol{\sigma}_n \text{ has exactly } \ell \text{ distinct } M\text{-griddings}\big]$$

be the asymptotic probability of having exactly ℓ griddings, and let

$$\kappa_M \;=\; \sum_{\ell\geqslant 1} P_\ell/\ell \;=\; \lim_{n\to\infty} \frac{\big|\mathsf{Grid}_n(M)\big|}{\big|\mathsf{Grid}_n^\#(M)\big|}$$

be the correction factor.

Then, $\big|\mathsf{Grid}_n(M)\big| \sim \kappa_M g^n \theta^\#(n)$.

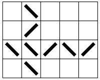
Non-corner peaks

A peak is a corner peak if one of its two cells is a corner.

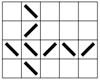
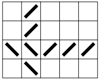
Dancing is always possible at every non-corner peak.

- Each non-corner peak doubles the number of griddings.
- If M has p non-corner peaks, then $\kappa_M = 2^{-p}\kappa_{M'}$, where M' is formed from M by removing the non-corner peaks.
  - The only dancing in $\mathsf{Grid}(M')$ is at the corners.

Example

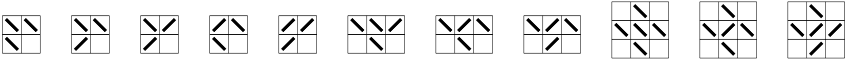


has 3 non-corner peaks, so

$\left|\mathsf{Grid}_n\left(\right.\right.$  $\left.\left.\right)\right| \;\sim\; \frac{1}{8} \times \left|\mathsf{Grid}_n\left(\right.\right.$  $\left.\left.\right)\right|.$

Corner types

There are 11 *inequivalent* corner types in connected one-corner classes:



L_0 L_1 L_3 L_4 L_7 T_2 T_4 T_5 X_0 X_4 X_7

Correction factors for corner types

τ		$\kappa(\tau)$	$\kappa(\tau^{\mathsf{R}})$
L_0		1	
L_1		$\frac{1}{2}\left(1+\frac{c\alpha\lambda}{\alpha+\gamma}\right)$	$\frac{1}{2}\left(1+\frac{r\alpha\lambda}{\alpha+\beta}\right)$
L_3		$\frac{\lambda\,(1-\lambda)}{(1-(c-1)\lambda)\,(1-(r-1)\lambda)}$	
L_7		$1-\lambda$	

- Multiply together to give correction factors for other corner types.

Summary of recipe

If M is connected and either acyclic or unicyclic:

- $\mathsf{Grid}^{\#}_n(M)$ can be enumerated (g.f. and asymptotics).
- Consider the uniform distribution over $\mathsf{Grid}^{\#}_n(M)$.
- Almost all gridded permutations have a close-to-optimal distribution of points between the cells, which can be calculated.
  - Gives limit shape.
- For almost all permutations, griddings are constrained to those that result from dancing.
- Analysis of possible dancing yields the asymptotic probability P_ℓ of an underlying permutation having exactly ℓ distinct griddings.
- $\left|\mathsf{Grid}_n(M)\right| \sim \kappa \left|\mathsf{Grid}^{\#}_n(M)\right|$, where $\kappa = \sum_{\ell \geqslant 1} P_\ell/\ell$.

Some references

- Michael Albert, M. D. Atkinson, Mathilde Bouvel, Nik Ruškuc, and Vincent Vatter (2013). **Geometric grid classes of permutations.** *Trans. Amer. Math. Soc.*, 365(11):5859–5881.
- Michael Albert and Vincent Vatter (2019). **An elementary proof of Bevan's theorem on the growth of grid classes of permutations.** *Proc. Edinburgh Math. Soc.*, 62(4):975–984.
- Noura Alshammari and David Bevan (2025). **On the asymptotic enumeration and limit shapes of monotone grid classes of permutations.** *DMTCS*, 27(1), Paper 2.
- M. D. Atkinson (1998). **Permutations which are the union of an increasing and a decreasing subsequence.** *Electron. J. Combin.*, 5: Paper R6.
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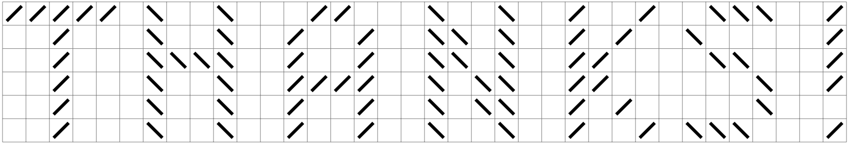
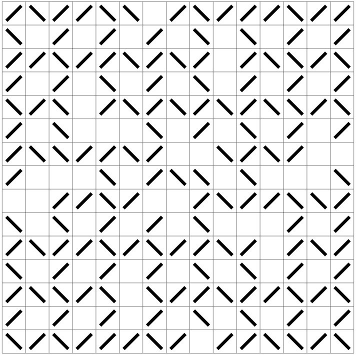
Thanks for listening!





Thanks for listening!