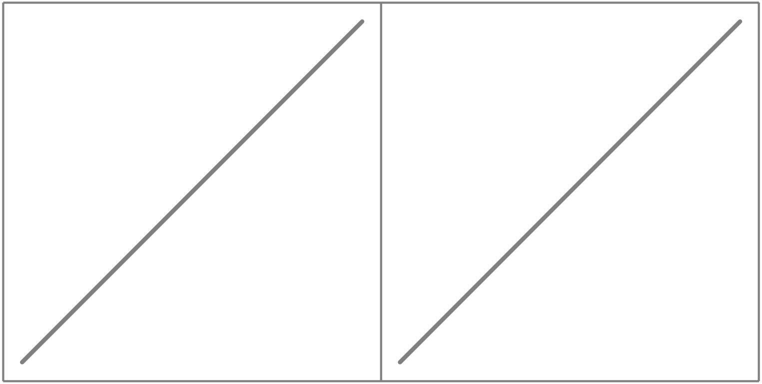
# Finitely based classes below 2.618 are rational

Robert Brignall
*Joint work with Michal Opler*

7th July 2025

The Open University

# The two cell class



This is $\mathrm{Av}(321, 3142, 2143)$, *but that's not important right now!*

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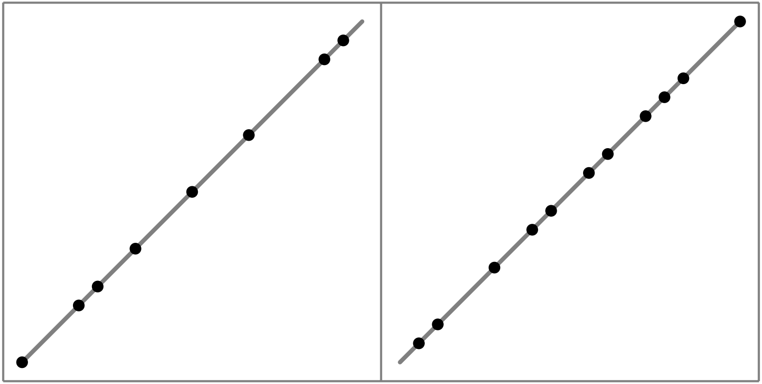

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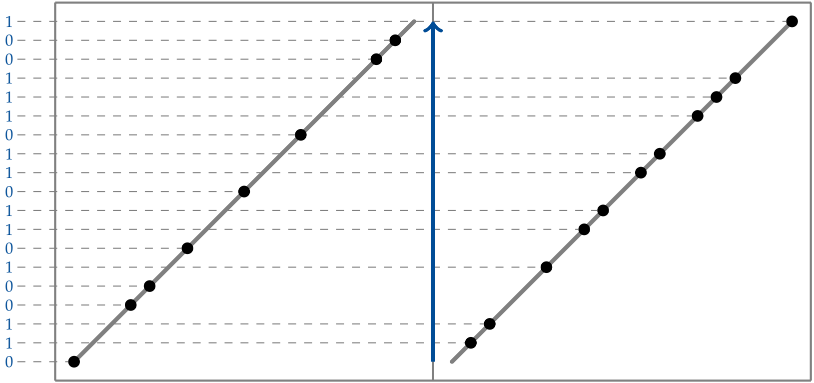
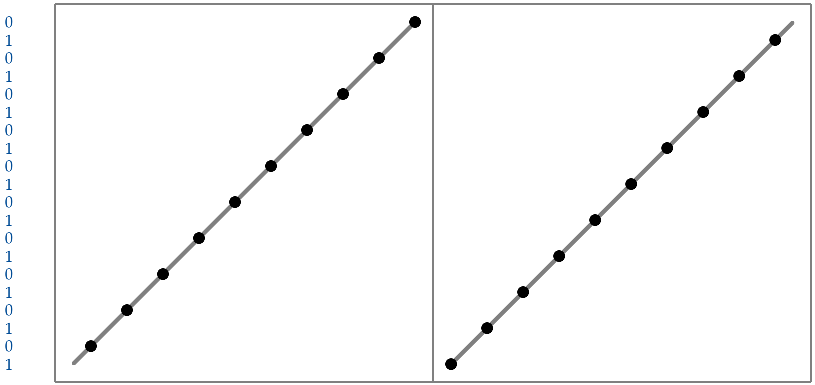



Encode to binary sequence. Gives us gr = 2.

# Alternations




Alternation: points alternate cells. Encodes as $1010\cdots$.

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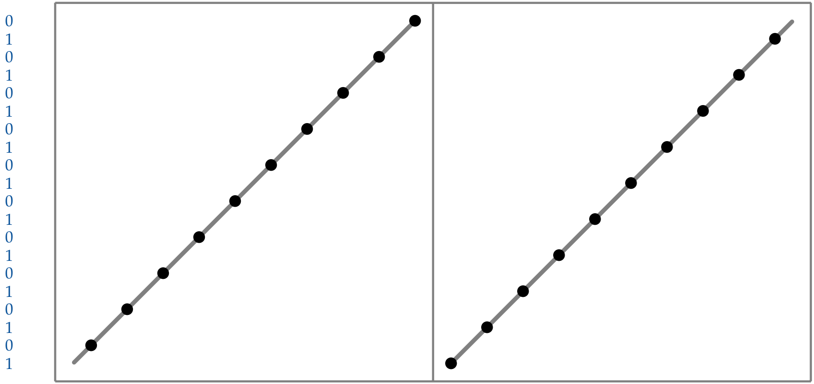


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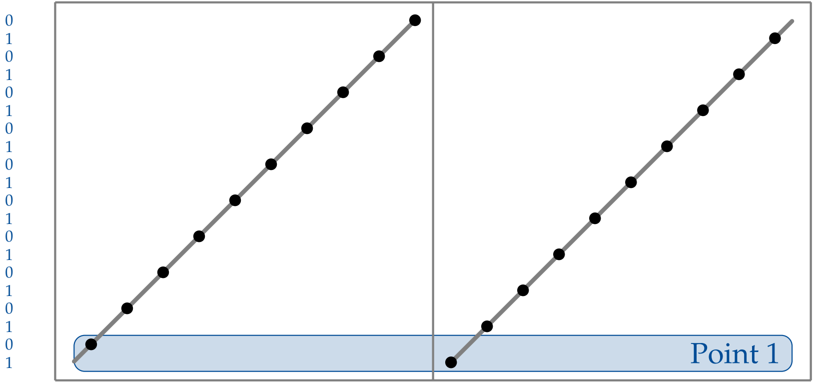


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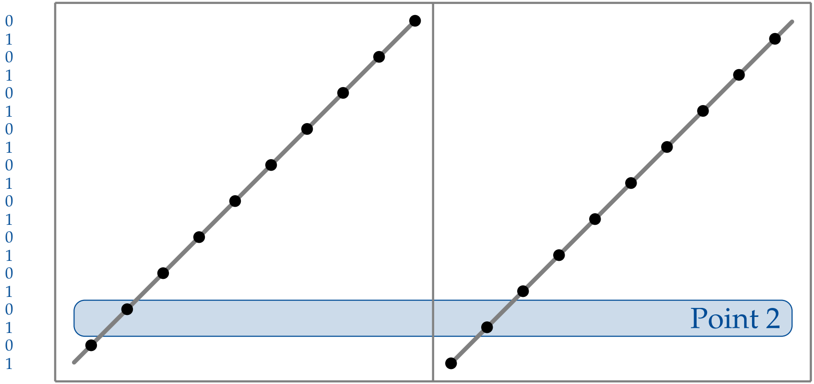


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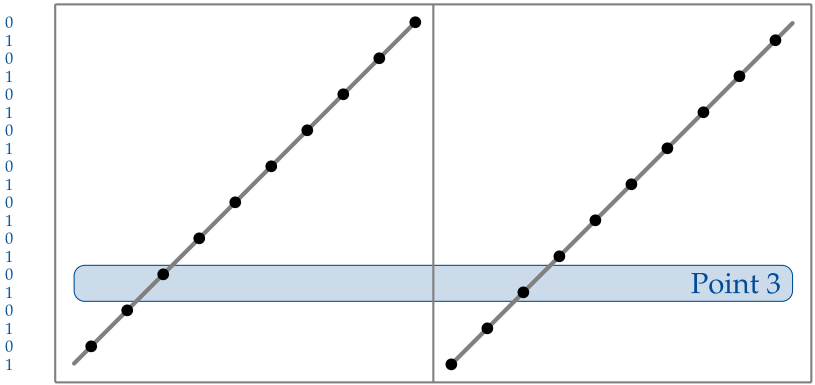


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# Permutation classes with gr $< 2$

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If a permutation class $\mathcal{C}$ contains long alternations, then $\mathrm{gr}(\mathcal{C}) \geqslant 2$.

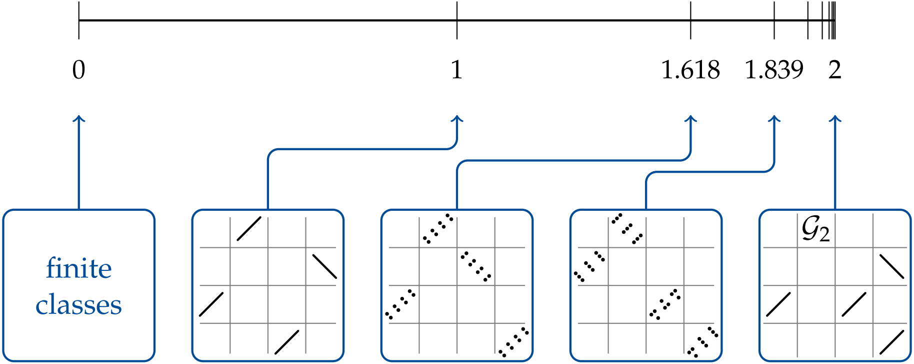
*conversely…*

If $\mathrm{gr}(\mathcal{C}) < 2$, then $\mathcal{C}$ contains only bounded length alternations.

# The Fibonacci dichotomy

Kaiser & Klazar (2002), Huczynska & Vatter (2006):

The only possible growth rates of permutation classes up to 2 are:



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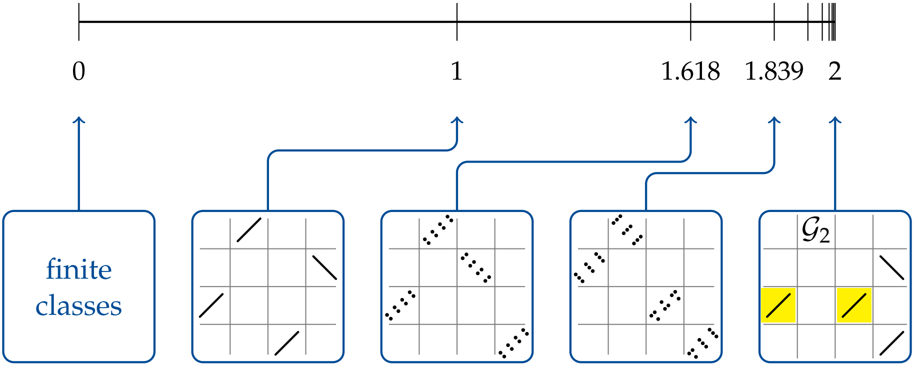

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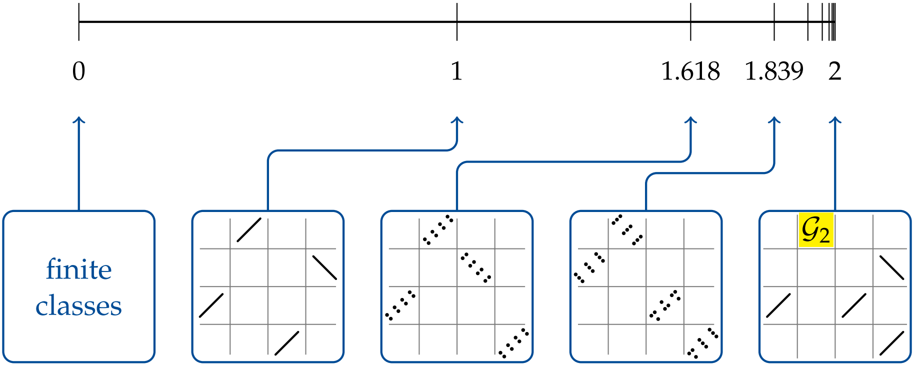
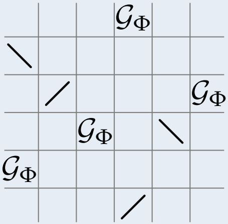



Isolated cells at growth rate = 2 can equal the cell class $\mathcal{G}_2$.

# Up to $\Phi = 1 + \phi \approx 2.618$

**Theorem (B., Opler (2025+))**

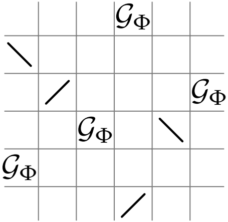
*Let $\mathcal{C}$ be a permutation class such that $\overline{gr}(\mathcal{C}) < \Phi$. Then there exists $K$ such that every $\pi \in \mathcal{C}$ is contained in a grid class of the form*




*except for at most $K$ points.*

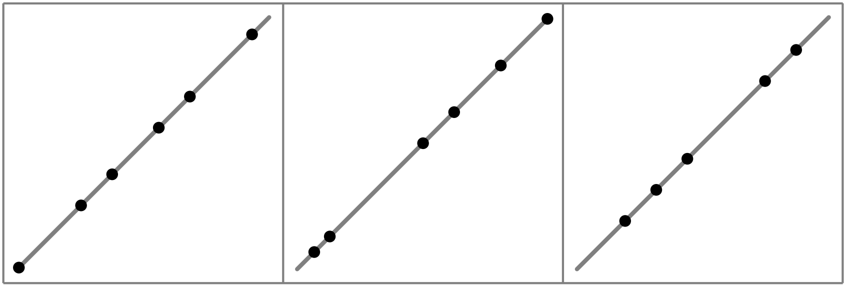
$\mathcal{G}_\Phi$ is the cell class at growth rate $\Phi$ (defined properly later!).

# Key features




1. Components have size 2.
2. Only three types of cell: $\mathcal{G}_\Phi$, monotone, or empty.
3. At most one nonmonotone entry per component.

# Three-cell classes I



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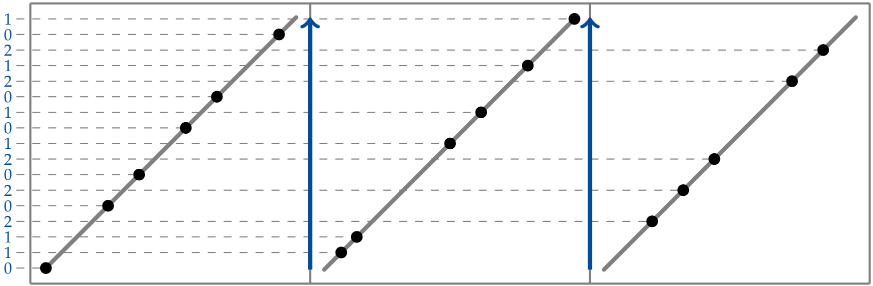


Encode using $\{0, 1, 2\}$. Gives us $\mathrm{gr} = 3$.

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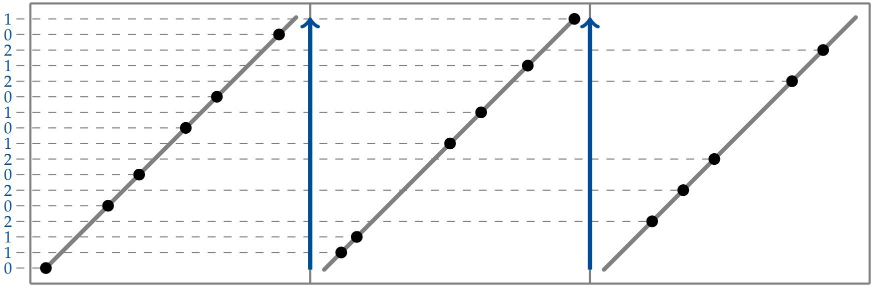


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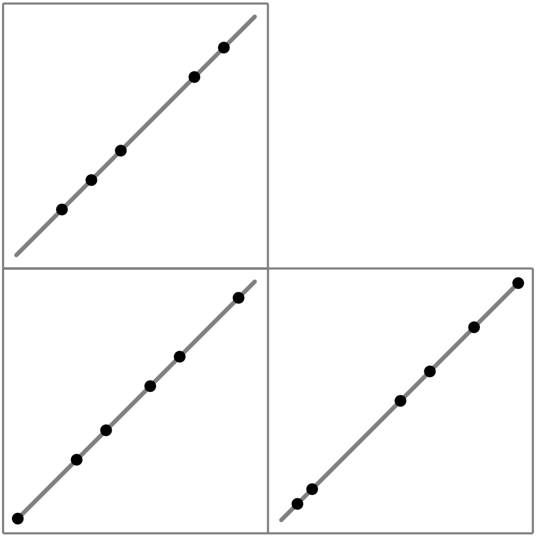
So if $\text{gr}(\mathcal{C}) < 3$, only bounded length 'triple alternations'.

# Three-cell classes II



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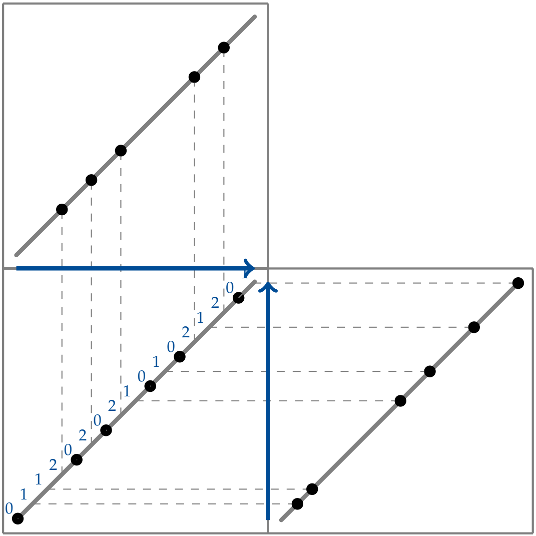


Encode using $\{0, 1, 2\}$: $\mathrm{gr} = 1 + \phi$.

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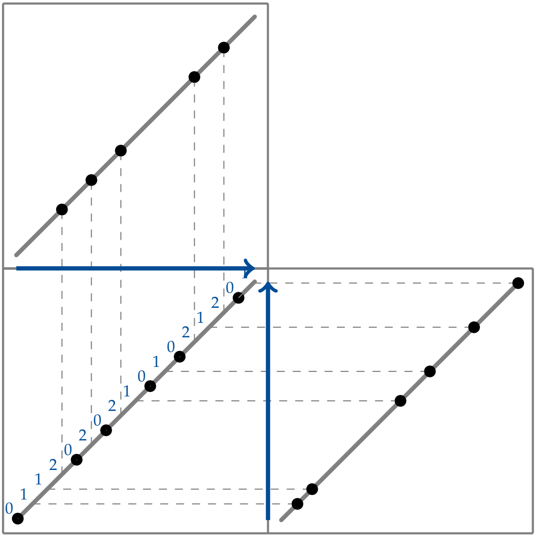
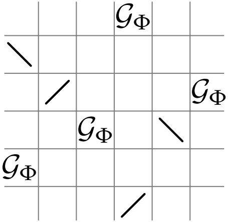



Encode using $\{0, 1, 2\}$: $\mathrm{gr} = 1 + \phi$. So no 'L-alternations' below 2.618.

# Key features

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# Cell classes

Following Vatter (2019): For real $\gamma > 0$, the cell class at $\gamma$ is

$$\mathcal{G}_\gamma = \left\{\pi \;:\; \mathrm{gr}(\mathrm{Sub} \bigoplus \pi) < \gamma \text{ or } \mathrm{gr}(\mathrm{Sub} \bigominus \pi) < \gamma\right\}.$$

**Example ($2413 \notin \mathcal{G}_2$)**

$\oplus$-indecomposables in 2413:

$$1, \quad 21, \quad 231, \quad 312, \quad 2413.$$

So $\mathrm{Sub} \bigoplus 2413$ has generating function

$$\frac{1}{1-(z+z^2+2z^3+z^4)}.$$

$\mathrm{gr}(\mathrm{Sub} \bigoplus 2413) = \frac{1}{\text{smallest real zero of denominator}} \approx 2.066.$

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## Theorem (Vatter (2019))

*If* $gr(\mathcal{C}) < \gamma$ *then, for some* $m \times n$ *array,*

$$\mathcal{C} \subseteq \begin{array}{c|c|c|c} \mathcal{G}_\gamma & \mathcal{G}_\gamma & \cdots & \mathcal{G}_\gamma \\ \hline \mathcal{G}_\gamma & \mathcal{G}_\gamma & & \mathcal{G}_\gamma \\ \hline \vdots & & \ddots & \vdots \\ \hline \mathcal{G}_\gamma & \mathcal{G}_\gamma & \cdots & \mathcal{G}_\gamma \end{array}$$

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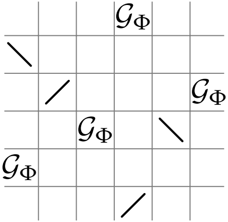
So when $\gamma = \Phi$, certainly each cell is a subclass of $\mathcal{G}_\Phi$.

# Key features

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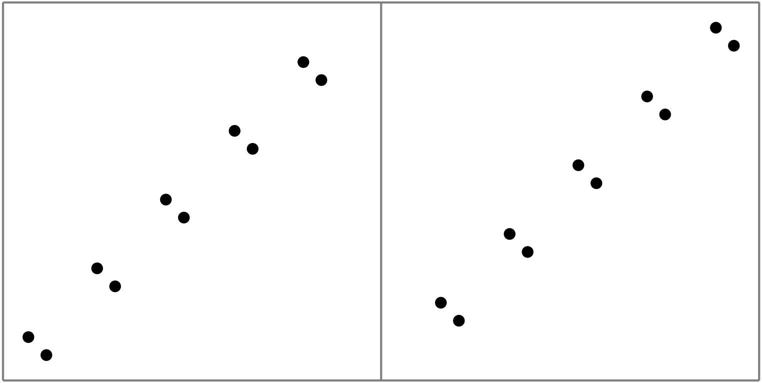
# Nonmonotone interactions

It turns out, the smallest class with two nonmonotone cells looks like:



Denominator of the enumeration is $1 - 2z - 2z^2$, giving

$$\mathrm{gr} = 1 + \sqrt{3} \approx 2.732.$$

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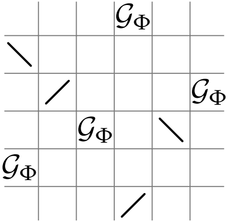


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# Actually proving it

We have to show:

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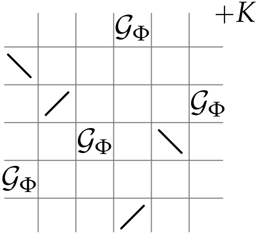
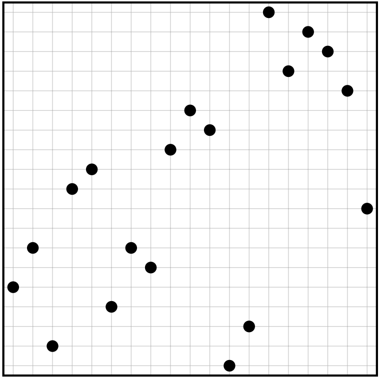



- Largely, proof follows Vatter (2019), with complications.
- For each $\pi \in \mathcal{C}$, systematically ‘slice’ rows and columns to eliminate structures that can’t exist.

# Concentration I

What happens when we slice a cell? For $\pi \in \mathcal{G}_\Phi$:



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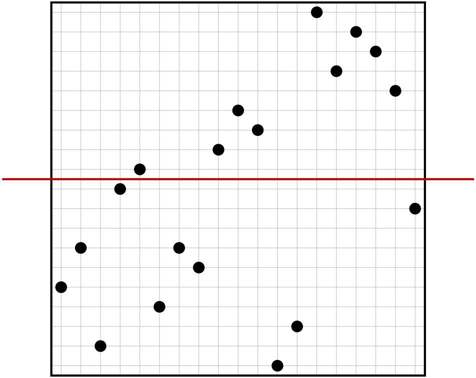

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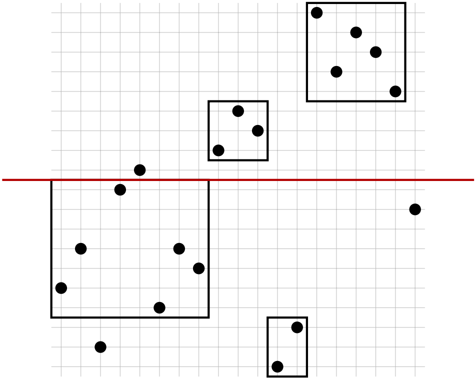

Draw (in this case) four noninteracting boxes.

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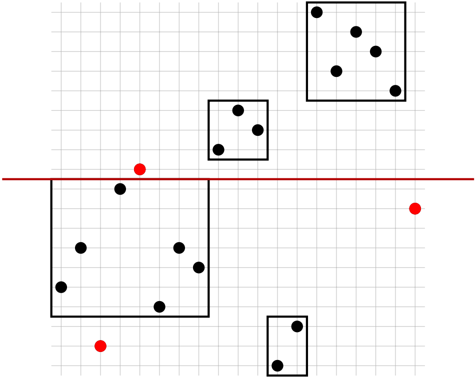

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Three points not in boxes.

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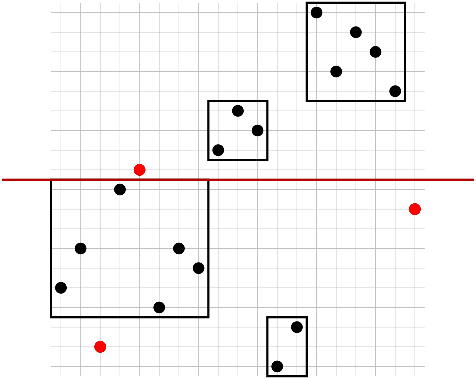

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Three points not in boxes.
This $\pi$ is $(4, 3)$-concentrated.

# Concentration II

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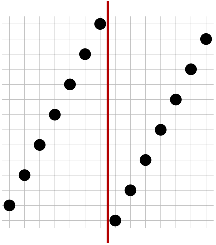
## Theorem (B., Opler (2025+))

*Yup. The cell class $\mathcal{G}_\gamma$ is concentrated if and only if $\gamma < \Phi$.*

An obvious barrier to concentration are long alternations:



These and other (worse) barriers do not appear in $\mathcal{G}_\gamma$ unless $\gamma \geqslant \Phi$.

## Theorem (B., Opler (2025+))

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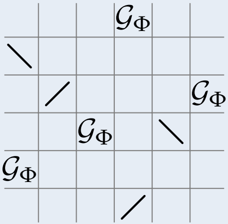


*except for at most $K$ points.*

*Now we use this to count the permutations in $\mathcal{C}$…*

# The insertion encoding

**Theorem (Albert, Linton, Ruškuc (2005))**

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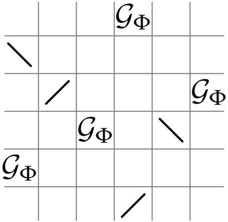


So each is individually amenable to the insertion encoding. We just need to stick it all together.

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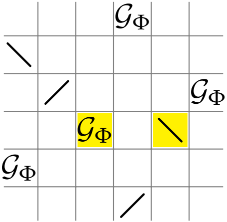


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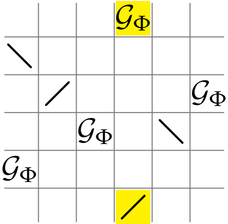


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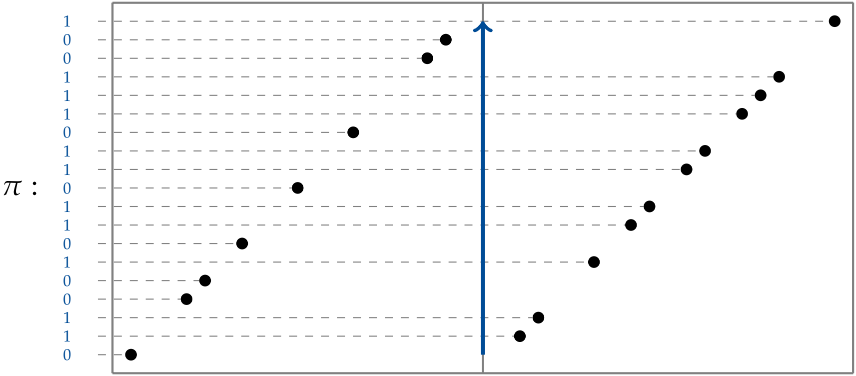
# MSO framework

The Büchi—Elgot—Trakhtenbrot Theorem:
A language is regular if and only if it is MSO definable.

> *"MSO may be seen as a high-level language that [. . . ] 'compiles' into an automaton, making it easier to describe complicated regular languages."*

Braunfeld (2024)

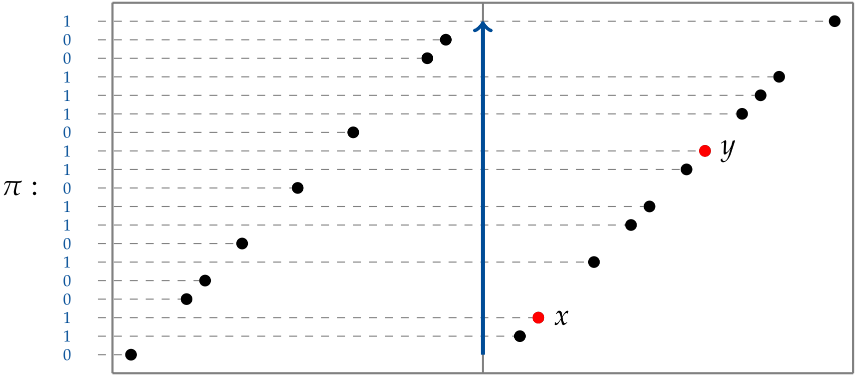
# An MSO teaser




$w$: 0 1 1 0 0 1 0 1 1 0 1 1 0 1 1 1 0 0 1

Each letter corresponds to a point in the permutation. How do two points relate?

# An MSO teaser

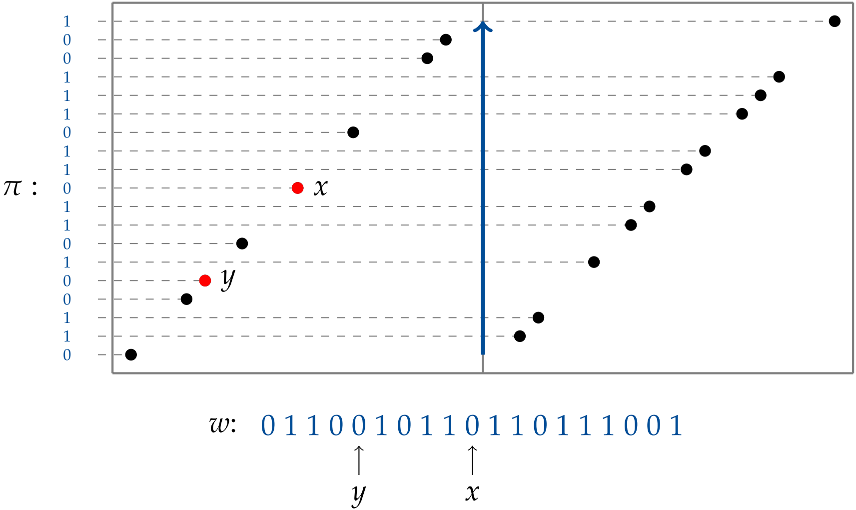



$w$: 0 1 1 0 0 1 0 1 1 0 1 1 0 1 1 1 0 0 1

$\uparrow$ $x$ $\qquad$ $\uparrow$ $y$

- $x$ is left of $y$ because they are both 1, and $x$ precedes $y$ in $w$.
- $x$ is below $y$ in $\pi$ because $x$ precedes $y$ in $w$.

# An MSO teaser




- $x$ is right of $y$ because they are both 0, and $y$ precedes $x$ in $w$.
- $x$ is above $y$ in $\pi$ because $y$ precedes $x$ in $w$.

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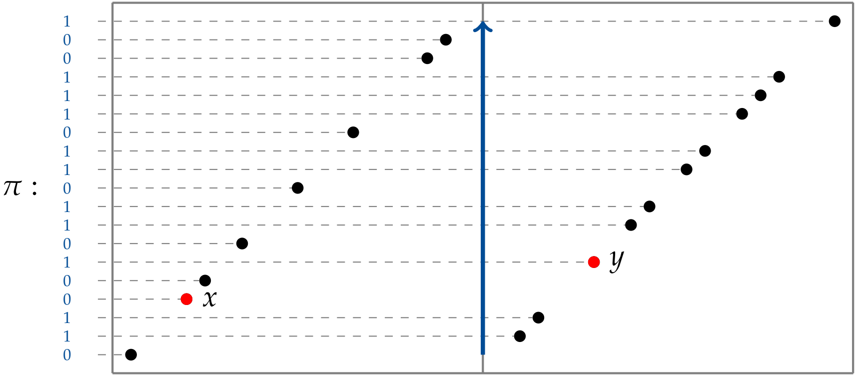
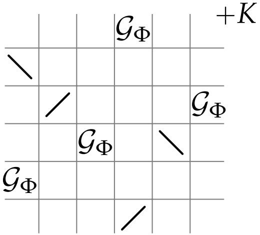



$w$: 0 1 1 0 0 1 0 1 1 0 1 1 0 1 1 1 0 0 1

↑ $x$ ↑ $y$

- $x$ is left of $y$ because $x$ is 0, and $y$ is 1.
- $x$ is below $y$ in $\pi$ because $x$ precedes $y$ in $w$.

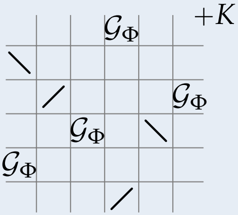
# Pull it all together




- Each component is insertion encodable ⇒ regular language ⇒ MSO encodable.
- If every component is MSO encodable, so is the whole class of *gridded* permutations.
- If the gridded permutations are MSO encodable, so are the ungridded permutations. [See, e.g., Braunfeld (2024)]
- If a class $\mathcal{D}$ is MSO encodable, so is $\mathcal{D}^{+K}$ for any fixed $K$.

## Theorem (B., Opler (2025+))

*Any MSO-definable subclass of*




*has a rational generating function.*

Since 'is finite based' is MSO definable…

## Corollary

*Every finitely based permutation class of growth rate* $< \Phi$ *has a rational generating function.*

# Thanks!


M. H. Albert, S. Linton, and N. Ruškuc. The insertion encoding of permutations.
*Electron. J. Combin.*, 12(1):Research paper 47, 31 pp., 2005

S. Braunfeld. Decidability in geometric grid classes of permutations.
*Proc. Amer. Math. Soc.*, 153(3):987–1000, 2025

S. Huczynska and V. Vatter. Grid classes and the Fibonacci dichotomy for restricted permutations.
*Electron. J. Combin.*, 13:Research paper 54, 14 pp. (electronic), 2006

T. Kaiser and M. Klazar. On growth rates of closed permutation classes.
*Electron. J. Combin.*, 9(2):Research paper 10, 20 pp. (electronic), 2003

V. Vatter. Growth rates of permutation classes: from countable to uncountable.
*Proc. Lond. Math. Soc. (3)*, 119(4):960–997, 2019