Wilf-equivalence of partial permutations

Alexander Burstein
Howard University
aburstein@howard.edu

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1. Definitions
2. Results
3. Conjectures

Partial permutations

Hamaker (PP'24): A partial permutation of size m is a pair of m-tuples of distinct positive integers (A, B), that is

$$A = (a_1, \dots, a_m), \quad B = (b_1, \dots, b_m).$$

A k-extension of a partial permutation (A, B) of size j is a permutation $\sigma \in S_k$ such that

$$\sigma(a_j) = b_j \quad \text{for each } j = 1, \dots, m.$$

This is defined for $k \geq m$.

A k-completion $(A, B)_k$ of a partial permutation (A, B) is the set of all k-extensions of (A, B).

For example,

- $((2,4),(1,2))_4 = \{3142, 4132\}$,
- $((2),(1))_4 = \{2134, 2143, 3124, 3142, 4123, 4132\}$,
- $(A, B)_k = \varnothing$ for $k < \max(A \cup B)$,
- Skip inside parentheses where context is non-ambiguous, e.g. $(24, 12)_3$, $(2, 1)_4$.

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Pattern avoidance

- A permutation π contains an occurrence (or instance) of pattern σ, there is a subsequence of π order-isomorphic to σ.
- π avoids σ if π does not contain an occurrence of σ.
- π avoids a set of patterns S if π avoids every pattern in $\sigma \in S$.
- Denote the set of permutations of size n avoiding a pattern σ (resp. a set of patterns S) by $\text{Av}_n(\sigma)$ (resp. by $\text{Av}_n(S)$).
- Call sets of patterns S and T Wilf-equivalent if $|\text{Av}_n(S)| = |Av_n(T)|$ for all $n \geq 0$, and denote this by $S \sim T$.

Wilf-equivalence for partial permutations

- Call partial permutations (A, B) and (C, D) k-Wilf-equivalent if $(A, B)_k \sim (C, D)_k$.
- Call partial permutations (A, B) and (C, D) Wilf-equivalent if $(A, B)_k \sim (C, D)_k$ for all $k \geq \max(A \cup B \cup C \cup D)$. Notation: $(A, B) \sim (C, D)$.
- For example, the following partial permutations are 3-Wilf-equivalent:

$$(13, 13)_3 = (12, 12)_3 \sim (13, 12)_3 \sim (23, 12)_3, \quad \text{i.e.} \quad 123 \sim 132 \sim 312,$$
$$(1, 1)_3 \sim (2, 1)_3 \sim (3, 1)_3, \quad \text{i.e.} \quad (123, 132) \sim (213, 312) \sim (231, 321).$$

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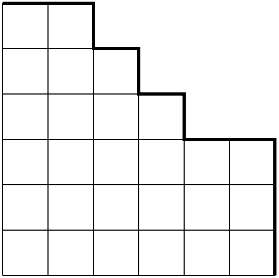
(NE)-shape-Wilf equivalence



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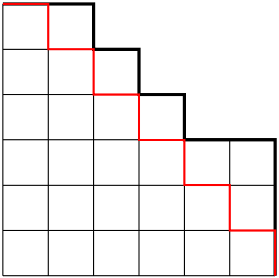

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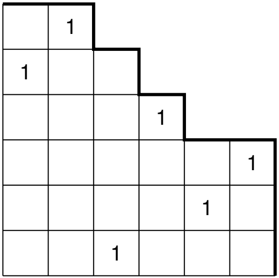

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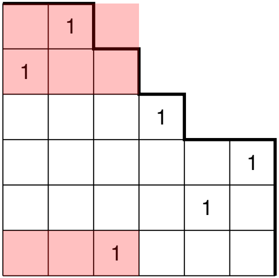

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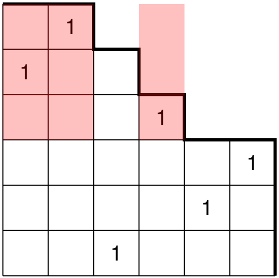

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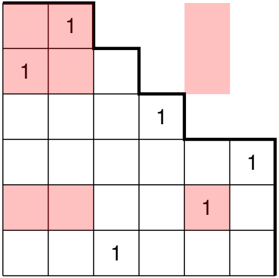

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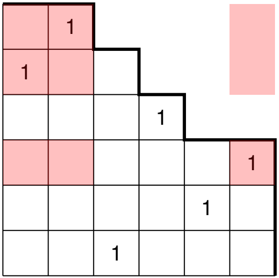

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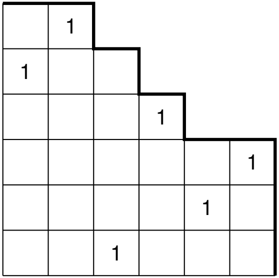

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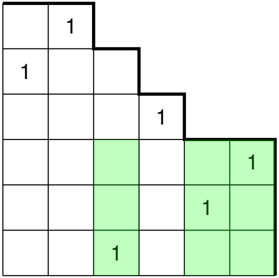

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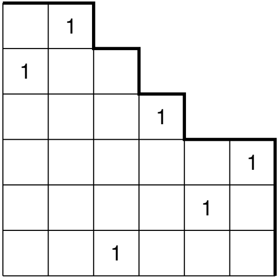

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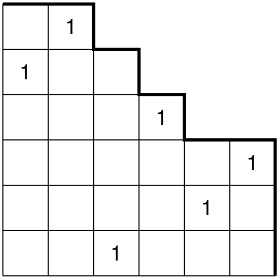

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- Patterns σ and τ are NE-shape-Wilf-equivalent, denoted $\sigma \sim_s \tau$, if for any fixed NE-shape Λ, equal number of traversals (or transversals) of Λ avoid σ and τ.

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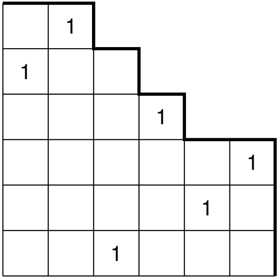

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- SW-shape-Wilf-equivalence can be defined similarly, denoted $\sigma \backsim_s \tau$

Previous results for NE-shape-Wilf equivalence

- Direct sum $\oplus$ and skew-sum $\ominus$:

- Backelin, West, Xin, 2007:
  - $S' \sim_s S'' \implies S' \oplus S \sim_s S'' \oplus S$, for any sets of patterns S, S', S''
  - Equivalently, $S' \backsim_s S'' \implies S \oplus S' \backsim_s S \oplus S''$, for any sets of patterns S, S', S''
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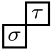
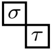
$\sigma \oplus \tau =$ 

$\sigma \ominus \tau =$ 

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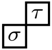
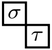
$\sigma \oplus \tau =$ 

$\sigma \ominus \tau =$ 

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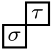
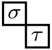
$\sigma \oplus \tau =$ 

$\sigma \ominus \tau =$ 

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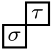
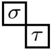
$\sigma \oplus \tau =$ 

$\sigma \ominus \tau =$ 

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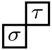
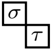
  $\sigma \oplus \tau =$ 

  $\sigma \ominus \tau =$ 

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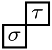
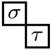
$\sigma \oplus \tau =$  $\sigma \ominus \tau =$ 

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  - $S' \sim_s S'' \implies S' \oplus S \sim_s S'' \oplus S$, for any sets of patterns S, S', S''
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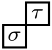
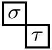
$\sigma \oplus \tau =$ 

$\sigma \ominus \tau =$ 

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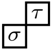
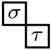
$\sigma \oplus \tau =$ 

$\sigma \ominus \tau =$ 

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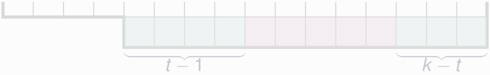
Theorem (B.-Han-Kitaev-Zhang, 2024)

$(t, 1)_k \backsim_s (1, 1)_k$ *for all* $1 \leq t \leq k$, *so all* $(t, 1)$, $t \geq 1$, *are SW-shape-Wilf-equivalent.*

Proof.

Grow the shape by adding the new bottom row and the column of the new bottom 1.



The number of possible insertion cells in a bottom row of length ℓ is $\min(\ell, k-1)$. □

This result was originally stated in terms of partially ordered patterns (POPs).

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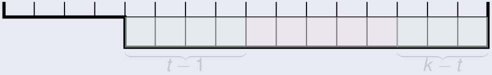

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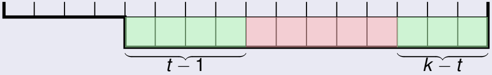

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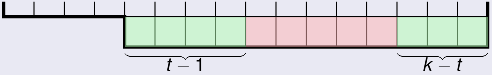

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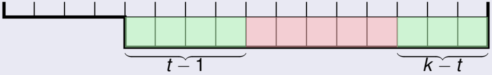

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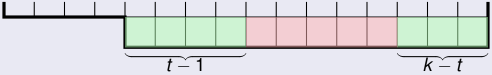

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Results II

Theorem

$((t, t+1), (1,2))_k \sim (12, 12)_k$ for all $1 \le t \le k-1$, so all $((t, t+1), (1,2))$, $t \ge 1$, are Wilf-equivalent.

Moreover, for $k \ge 3$, $$|\operatorname{Av}_n(((t, t+1), (1,2))_k)| = \begin{cases} n!, & \text{if } n < k-3, \\ (k-3)!\, r_{k-3}(n), & \text{if } n \ge k-3, \end{cases}$$

where $r_{k-3}(n)$ is the n-th $(k-3)$-Schröder number, the number of Schröder paths from $(0,0)$ to $(2n, 0)$ on or above the x-axis with steps $U = (1,1)$, $D = (1,-1)$, and steps $H = (2,0)$ of $k-3$ colors.

Proof Sketch.

For $\sigma \in \operatorname{Av}_n(((t, t+1), (1,2))_k)$, consider the top $k-2$ values of σ (their order is irrelevant) and the blocks into which they split σ. Count the number of the points in the region that avoids 12. Use this to find a functional equation for the generating function with 1 auxiliary variable, then use the kernel method. □

Proof Sketch (cont'd)

Let $i = t - 1$, $j = k - t - 1$ (so $k - 2 = i + j$), and let $R_{ij} = ((i+1, i+2), (1,2))_{i+j+2}$. Note that $R_{00} = (12, 12)_2 = 12$.

$$F(x,y) = 1 + (k-2)x\frac{yF(x,y) - F(x,1)}{y-1} + xyF(x,y)$$

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Kernel $= 0$: $y = 1 + (k-3)xy + xy^2$, i.e. $y = R_{k-3}(x)$, the $(k-3)$-Schröder ogf.

OGF over $j \geq k-3$:

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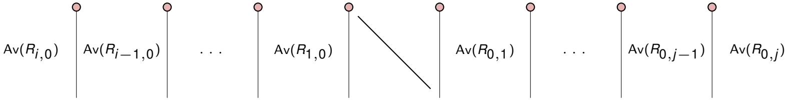

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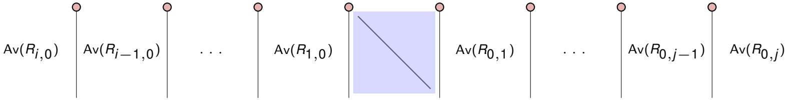

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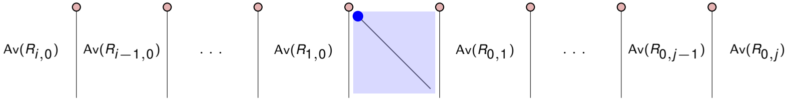

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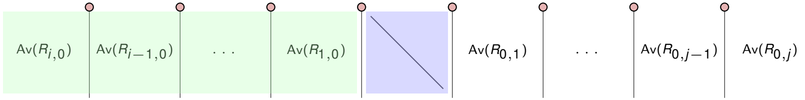

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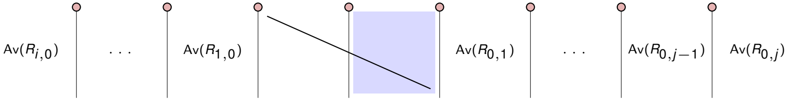

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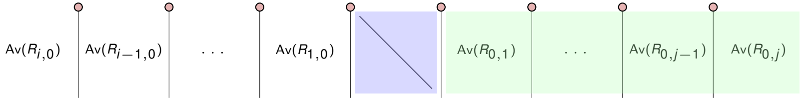

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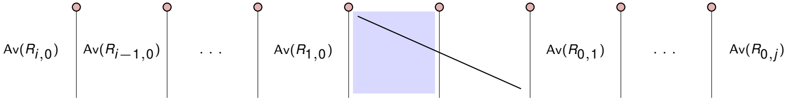

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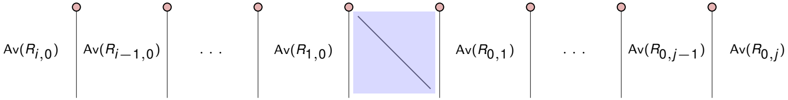

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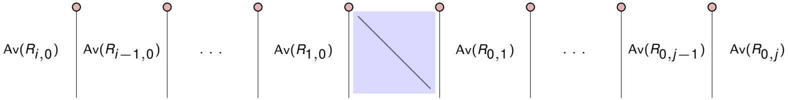

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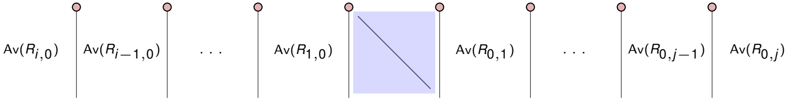
Let $i = t - 1$, $j = k - t - 1$ (so $k - 2 = i + j$), and let $R_{ij} = ((i+1, i+2), (1,2))_{i+j+2}$. Note that $R_{00} = (12, 12)_2 = 12$.



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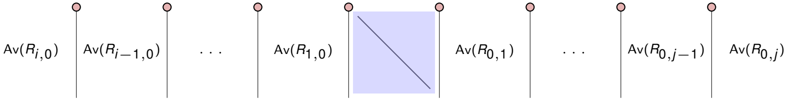

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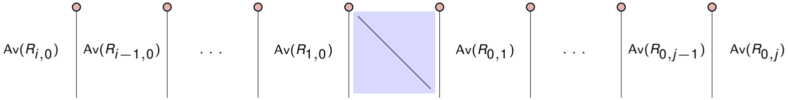

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- $(ij, 12)$ are Wilf-equivalent for all (i, j) such that $i = 1$ or $j = i \pm 1$ or $j = i \pm 2$.
- (BHKZ'24): $(k, 2)_k \backsim_s (k, 1)_k$, and thus $(1k, 13)_k \backsim_s (1k, 12)_k$ for all $k \geq 3$.
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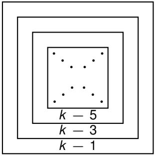

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We know that $|Av_n((t,1)_k)| = (k-1)!(k-1)^{n-k+1}$ for $1 \leq t \leq k$.

Here are more examples for (i,j) with $2 \leq i,j \leq k-1$.

- $|Av_n((2,2)_4)| = \text{A128445}(n) = 4((n-2)^2+1)$ for $n \geq 5$.
- $|Av_n((2,2)_5)| = \frac{9}{8}\text{A217527}(n) = \frac{9}{4}\text{A356888}(n-1) = 9((n-2)^2+2)2^{n-5}$ for $n \geq 6$.
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