On inversions, major index, and the displacement statistic on ℓ-interval parking functions

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July 11, 2025

Arxiv 2507.07243

Graduate Research Workshop in Combinatorics

This research started at Graduate Research Workshop in Combinatorics 2024 held at UWisconsin-Milwaukee (funded by the NSF Grant No. DMS – 1953445).

Figure: Statistics in Parking Functions group at GRWC 2024

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Parking functions (Konheim–Weiss)

A *parking function* of length n is a word $\alpha = \alpha_1 \cdots \alpha_n$ of positive integers such that if $\beta = \beta_1 \cdots \beta_n$ is α sorted weakly increasingly then $\beta_i \leq i$ for all i.

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Suppose we have n cars that want to park in spots $1, 2, \ldots, n$ on a 1 way street, with car i desiring spot α_i.

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Suppose we have n cars that want to park in spots $1, 2, \ldots, n$ on a 1 way street, with car i desiring spot α_i.The cars park 1 by 1:

- car i takes spot α_i, if available
- if not, then car i takes spot $\alpha_i + 1$, if available
- if $\alpha_i + 1$ is also taken, then it tries to park in spot $\alpha_i + 2$
- and so on.

If it runs out of spots, then car i cannot park. If all cars *do* park, then $\alpha = (\alpha_1, \ldots, \alpha_n)$ is called a *parking function*.

$\alpha = 22131$

car	1	2	3	4	5
parking spot	2	3	1	4	5

A few basic facts about parking functions

Let PF_n be the set of parking functions of length n

$$\mathrm{PF}_3 = \begin{matrix} 123 & 122 & 113 & 112 & 111 \\ 132 & 212 & 131 & 121 & \\ 213 & 221 & 311 & 211 & \\ 231 & & & & \\ 312 & & & & \\ 321 & & & & \end{matrix}$$

1. $|\,\mathrm{PF}_n\,| = (n+1)^{n-1}$ (Konheim–Weiss '66).
2. Any rearrangement of a parking function is a parking function.
3. There are $C_n = \frac{1}{n+1}\binom{2n}{n}$ many weakly increasing parking functions.
4. Every permutation is a parking function.

ℓ-interval parking functions

The *displacement* of car i is how far away car i must park from its desired spot. Let *maxdisp*(α) be the maximum displacement of a car of α

$\alpha = 22131$

car	1	2	3	4	5
parking spot	2	3	1	4	5
displacement	0	1	0	1	4

α is an ℓ*-interval parking function* if *maxdisp*$(\alpha) \leq \ell$.

maxdisp							
0	123	132	213	231	312	321	
1	122	212	221	113	131	313	112
2	121	211	111				

Let $\mathrm{IPF}_n(\ell)$ be the set of ℓ-interval parking functions of length n. E.g. $\mathrm{IPF}_n(n-1) = \mathrm{PF}_n$ and $\mathrm{IPF}_n(0) = \mathfrak{S}_n$.

Goals

Let $\mathrm{IPF}_n(\ell)$ be the set of ℓ-interval parking functions of length n.

1. Find formulas for $|\mathrm{IPF}_n(\ell)|$
   - $|\mathrm{IPF}_n(0)| = |\mathfrak{S}_n| = n!$ and $|\mathrm{IPF}_n(n-1)| = |\mathrm{PF}_n| = (n+1)^{n-1}$
2. Study permutation statistics on $\mathrm{IPF}_n(\ell)$.
   - inv and maj equidistribution?
   - They equidsitributed on $\mathfrak{S}_n = \mathrm{IPF}_n(0)$ and $\mathrm{PF}_n = \mathrm{IPF}_n(n-1)$

Counting by permutations

Spots permutation $\text{spot}(\alpha) = \sigma_1 \cdots \sigma_n$, where σ_i is the the spot that car i parked in.

α	2	2	1	3	1
$\text{spot}(\alpha)_i$	2	3	1	4	5
$\text{disp}(\alpha)_i$	0	1	0	1	4

Note that $\text{disp}(\alpha)_i = \text{spot}(\alpha)_i - \alpha_i$

$L_\ell(\sigma; i) := \min(\ell + 1, \sigma_i - t + 1)$, where t is the smallest number such that $\sigma^{-1}(j) \leq i$ for all $t \leq j \leq \sigma_i$

σ	2	3	1	4	5
$L_0(\sigma; i)$	1	1	1	1	1
$L_1(\sigma; i)$	1	2	1	2	2
$L_2(\sigma; i)$	1	2	1	3	3
$L_3(\sigma; i)$	1	2	1	4	4
$L_4(\sigma; i)$	1	2	1	4	5

Counting by permutations

Proposition (CEHHMPU ('25))

$$|\operatorname{IPF}_n(\ell)| = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^{\ell} L_\ell(\sigma; i)$$

q-Counting by permutations

Let $\mathrm{disp}(\alpha) = \sum_i \mathrm{disp}_i(\alpha)$ and $[n]_q = 1 + q + \cdots + q^{n-1}$.

Proposition (CEHHMPU ('25))

$$\sum_{\alpha \in \mathrm{IPF}_n(\ell)} q^{\mathrm{disp}(\alpha)} = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^{\ell} [L_\ell(\sigma; i)]_q$$

qt-Counting by permutations

Let $\mathrm{disp}(\alpha) = \sum_i \mathrm{disp}_i(\alpha)$ and $[n]_q = 1 + q + \cdots + q^{n-1}$. Let $\mathrm{inv}(\alpha)$ be the number of inversions of α

Theorem (CEHHMPU ('25))

$$\sum_{\alpha \in \mathrm{IPF}_n(\ell)} q^{\mathrm{disp}(\alpha)} t^{\mathrm{inv}(\mathrm{spot}(\alpha))} = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{inv}(\sigma)} \prod_{i=1}^{\ell} [L_\ell(\sigma; i)]_q$$

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Theorem (CEHHMPU ('25))

$$\sum_{\alpha \in \mathrm{IPF}_n(\ell)} q^{\mathrm{disp}(\alpha)} t^{\mathrm{stat}(\mathrm{spot}(\alpha))} = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{stat}(\sigma)} \prod_{i=1}^{\ell} [L_\ell(\sigma; i)]_q$$

Note: inv can be replaced by any permutation statistic stat.

qt-Counting $\mathrm{IPF}_n(n-2)$ in terms of $\mathrm{IPF}_n(n-1)$

Let $\ell = n-2$. Recall $\mathrm{PF}_n = \mathrm{IPF}_n(n-1)$.

Corollary (CEHHMPU ('25))

$$\sum_{\alpha \in \mathrm{IPF}_n(n-2)} q^{\mathrm{disp}(\alpha)} t^{\mathrm{inv}(\alpha)} = \sum_{\alpha \in \mathrm{PF}_n} q^{\mathrm{disp}(\alpha)} t^{\mathrm{inv}(\alpha)} - (qt)^{n-1} \sum_{\beta \in \mathrm{PF}_{n-1}} q^{\mathrm{disp}(\beta)} t^{\mathrm{inv}(\beta) - ones(\beta)},$$

where $ones(\beta) = |\{i \in [n-1] \mid \beta_i = 1\}|$.

In particular,

$$|\,\mathrm{IPF}_n(n-2)| = (n+1)^{n-1} - n^{n-2}$$

qt-Counting $\mathrm{IPF}_n(1)$ in terms of $\mathrm{IPF}_n(0)$

Let $\ell = 1$. Recall $\mathfrak{S}_n = \mathrm{IPF}_n(0)$.

Corollary (CEHHMPU ('25))

$$\sum_{\alpha \in \mathrm{IPF}_n(1)} q^{\mathrm{disp}(\alpha)} t^{\mathrm{inv}(\alpha)} = \sum_{\sigma \in \mathfrak{S}_n} (1+q)^{\mathrm{asc}(\sigma^{-1})} t^{\mathrm{inv}(\sigma)}$$

In particular,

$$|\,\mathrm{IPF}_n(1)| = \sum_{\sigma \in \mathfrak{S}_n} 2^{\mathrm{asc}(\sigma)} = \mathrm{Fub}_n$$

where Fub_n is the n-th Fubini number aka ordered Bell number aka number of ordered set partitions.

OSP structure of $\mathrm{IPF}_n(1)$ (Bradt et al '24)

The bijection $\mathrm{IPF}_n(1) \to \mathrm{OSP}_n$ is:

$\alpha = 81551247$

$\beta = 112|4|55|7|8$

$B(\alpha) = 256/7/34/8/1$

1. Let β be $\alpha \in \mathrm{IPF}_n(1)$ in increasing order
2. Put separators before each pos. i such that $\beta_i = i$.
3. The pos. of elem's of each block forms the OSP $B(\alpha)$

Lemma

$\mathrm{inv}(\alpha) = \#\{(i,j) \mid i > j \textit{ and } i \textit{ is in earlier block of } B \textit{ than } j\}$

$\mathrm{disp}(\alpha) = n - (\# \textit{ of blocks of } \pi)$

Consequences of OSP structure (**C**EHHMPU ('25))

Corollary ($q = -1$)

$\sum_{\alpha \in \mathsf{IPF}_n(1)} (-1)^{\mathsf{disp}(\alpha)} t^{\mathsf{inv}(\alpha)} = t^{\binom{n}{2}}$.

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Proof 1 (Algebraic).

$\sum_{\alpha \in \mathsf{IPF}_n(1)} (-1)^{\mathsf{disp}(\alpha)} t^{\mathsf{inv}(\alpha)} = \sum_{\sigma \in \mathfrak{S}_n} (1-1)^{\mathsf{asc}(\sigma^{-1})} t^{\mathsf{inv}(\sigma)}.$ □

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Proof 2 (Combinatorial).

Sign-reversing involution: Let $B(\alpha) = B_1/B_2/\cdots$. Select min. k s.t. either (i) $|B_k| \geq 2$, or (ii) $|\pi_k| = 1$ and $\min \pi_k < \min \pi_{k+1}$. In (i), break B_k at first element. In (ii) merge B_k and B_{k+1}.

Can do this for every α except $\alpha = w_0$ (longest permutation). □

$81551247 \mapsto 256/7/34/8/1 \mapsto 2/56/7/34/8/1 \mapsto 81552247$

Consequences of OSP structure (**C**EHHMPU ('25))

Corollary ($t = \omega = e^{2\pi i/n}$)

If $\omega = e^{2\pi i/n}$, then $\sum_{\alpha \in \mathsf{IPF}_n(1)} q^{\mathsf{disp}(\alpha)} \omega^{\mathsf{inv}(\alpha)} = q^{n-1}$.

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Then do some computations involving $\left[{n \atop c_1, \ldots, c_\ell} \right]_{t=\omega}$. □

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Proof 2 (Combinatorial).

OSP_n has $C_n = \langle c \mid c^n = 1 \rangle$-action w/ fixed pt $B_e = 12 \cdots n$. □

Proof 2 generalizes to show a (graded) "cyclic sieving"

$$\sum_{\alpha \in \mathrm{IPF}_n(1)} q^{\mathrm{disp}(\alpha)} (\omega^j)^{\mathrm{inv}(\alpha)} = \sum_{\substack{\pi \in \mathrm{OSP}_n \\ c^j \cdot \pi = \pi}} q^{n - blocks(\pi)}.$$

Inversion sequences for $\mathrm{IPF}_n(1)$

$\alpha = 81515247$

$\beta = 112|4|55|7|8$

$L(\alpha) = 000|0|32|0|7$

Proposition

$\mathrm{IPF}_n(1)$ *is in bijection with the set of pairs* (w, S), *where* $w \in \{0\} \times \{0,1\} \times \cdots \times \{0,1,\ldots,n-1\}$ *and* $S \supseteq \mathrm{ASC}(w)$. *Moreover, if* $\alpha \mapsto (w, S)$ *then* $\mathrm{inv}(\alpha) = \sum_{i=1}^{n} w_i$ *and* $\mathrm{disp}(\alpha) = n - |S|$.

Ex. $(w, S) = 0000|320|7 \mapsto \alpha' = 81515236$

qt-Counting $\mathrm{IPF}_n(2)$ in terms of $\mathrm{IPF}_n(1)$

$\beta \in \mathrm{IPF}_n(1)$. $B = B(\beta) \in \mathrm{OSP}_n$.
$\mathcal{S}(\beta) := \sum_{i \geq 1} \max(|B_i| - 2, 0)$
$\mathcal{R}(\beta) := \#\{B_i \mid B_i(2) > \max(B_{i-1})\}$.

Theorem (CEHHMPU ('25))

$$\sum_{\alpha \in \mathrm{IPF}_n(2)} q^{\mathrm{disp}(\alpha)} t^{\mathrm{inv}(\alpha)} = \sum_{\beta \in \mathrm{IPF}_n(1)} q^{\mathrm{disp}(\beta)} t^{\mathrm{inv}(\beta)} (1+q)^{\mathcal{S}(\beta)} (1+qt)^{\mathcal{R}(\beta)}.$$

In particular, $|\mathrm{IPF}_n(2)| = \sum_{\beta \in \mathrm{IPF}_n(1)} 2^{\mathcal{S}(\beta)+\mathcal{R}(\beta)}$.

Equidistribution

Recall that inv and maj are equidistributed on $\mathfrak{S}_n$ i.e.

$$\sum_{\sigma \in \mathfrak{S}_n} t^{\mathsf{inv}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathsf{maj}(\sigma)}$$

More generally, equidistribution holds for any set of words that is closed under rearrangements (aka $\mathfrak{S}_n$-invariant). Hence, equidistribution holds for $\mathsf{IPF}_n(n-1) = \mathsf{PF}_n$.

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Question

For what ℓ are inv and maj equidistributed on $\mathrm{IPF}_n(\ell)$?

Equidistribution

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But other $\mathrm{IPF}_n(\ell)$ are not $\mathfrak{S}_n$-invariant.

$$\begin{array}{cccc} \alpha & 1 & 1 & 2 \\ \mathrm{disp}(\alpha)_i & 0 & 1 & 1 \end{array} \in \mathrm{IPF}_3(1) \quad \text{but} \quad \begin{array}{cccc} \beta & 1 & 2 & 1 \\ \mathrm{disp}(\beta)_i & 0 & 0 & 2 \end{array} \notin \mathrm{IPF}_3(1)$$

In fact, for ALL $0 < \ell < n-1$, $\mathrm{IPF}_n(\ell)$ is not $\mathfrak{S}_n$-invariant.

Foata transformation

Recall the Foata transformation $F : \mathbb{Z}^*_{>0} \to \mathbb{Z}^*_{>0}$, which is a bijection such that $\text{maj}(w) = \text{inv}(F(w))$ for words w

$w = 6144512$

$$6$$
$$6|1$$
$$61|4 \to 164$$
$$1|64|4 \to 1464$$
$$1|4|64|5 \to 14465$$
$$14|4|6|5|1 \to 414651$$
$$41|4651|2 \to 1414652 = F(w)$$

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$\alpha = 6144512 \in \mathrm{IPF}_7(2)$

$$
\begin{aligned}
&6\\
&6|1\\
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&41|4651|2 \to 1414652 = F(\alpha) \in \mathrm{IPF}_7(2)
\end{aligned}
$$

Foata invariance

Theorem (CEHHMPU ('25))

- *For* $\ell \in \{0, 1, 2, n-2, n-1\}$*, the Foata transformation restricts to a bijection* $\mathsf{IPF}_n(\ell) \to \mathsf{IPF}_n(\ell)$*, and consequently the* inv *and* maj *are equidistributed on* $\mathsf{IPF}_n(\ell)$*.*
- *For* $2 < \ell < n-2$*,* inv *and* maj *are not equidistributed on* $\mathsf{IPF}_n(\ell)$*.*

In fact, for all $2 < \ell < n-2$ we can show

$$\#\{\alpha \in \mathsf{IPF}_n(\ell) \mid \mathsf{inv}(\alpha) = 1\} > \#\{\alpha \in \mathsf{IPF}_n(\ell) \mid \mathsf{maj}(\alpha) = 1\}$$

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Note that $\{0, 1, 2, n-2, n-1\}$ is the same set of ℓ for which we have nice enumerative formulas. Coincidence???

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Note that $\{0, 1, 2, n-2, n-1\}$ is the same set of ℓ for which we

have nice enumerative formulas. Coincidence??? Probably, but still useful!

Foata invariance

Lemma

$\operatorname{disp}(F(\alpha)) = \operatorname{disp}(\alpha)$ *for all* $\alpha \in \mathrm{PF}_n$.

Corollary (**C**EHHMPU ('25))

$$\sum_{\alpha \in \mathrm{IPF}_n(n-2)} q^{\operatorname{disp}(\alpha)} t^{\operatorname{maj}(\alpha)} = \sum_{\alpha \in \mathrm{PF}_n} q^{\operatorname{disp}(\alpha)} t^{\operatorname{maj}(\alpha)} - (qt)^{n-1} \sum_{\beta \in \mathrm{PF}_{n-1}} q^{\operatorname{disp}(\beta)} t^{\operatorname{maj}(\beta) - ones(\beta)}$$

$$\sum_{\alpha \in \mathrm{IPF}_n(1)} q^{\operatorname{disp}(\alpha)} t^{\operatorname{maj}(\alpha)} = \sum_{\sigma \in \mathfrak{S}_n} (1+q)^{\operatorname{asc}(\sigma^{-1})} t^{\operatorname{maj}(\sigma)}.$$

Note: $ones(F(\beta)) = ones(\beta)$ and $\operatorname{asc}(F(\sigma)^{-1}) = \operatorname{asc}(\sigma^{-1})$

Foata invariance

Lemma

$\mathcal{R}(F(\beta)) = \mathcal{R}(\beta)$ and $\mathcal{S}(F(\beta)) = \mathcal{S}(\beta)$ for all $\beta \in \mathrm{IPF}_n(1)$.

Corollary (CEHHMPU ('25))

$$\sum_{\alpha \in \mathrm{IPF}_n(2)} q^{\mathrm{disp}(\alpha)} t^{\mathrm{maj}(\alpha)} = \sum_{\beta \in \mathrm{IPF}_n(1)} q^{\mathrm{disp}(\beta)} t^{\mathrm{maj}(\beta)} (1+q)^{|\mathcal{S}(\beta)|} (1+qt)^{|\mathcal{R}(\beta)|}.$$

Note: The only proof we know of these formulas are combining the previous results with the Foata transformation! A direct proof is unknown.

Open problems

1. Find formulas for $\mathrm{IPF}_n(\ell)$ in terms of $\mathrm{IPF}_n(\ell \pm 1)$.
   - We conjecture that $(|\mathrm{IPF}_n(\ell) \setminus \mathrm{IPF}_n(\ell - 1)|)_{\ell=0}^{n-1}$ is unimodal.
2. Describe a sort of "OSP structure" for $\mathrm{IPF}_n(2)$
3. Determine the relation between $F(\mathrm{spot}(\alpha))$ and $\mathrm{spot}(F(\alpha))$
   - For $\alpha \in \mathrm{IPF}_n(1)$, we conjecture that $F(\mathrm{spot}(\alpha)) = \mathrm{spot}(F(\alpha))$
   - Equality does not hold in general.
4. Explore pattern avoidance for $\mathrm{IPF}_n(1)$
   - Its OSP structure should come in handy.

Thanks!

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