Descents of permutations with only even or only odd cycles

Sergi Elizalde

Dartmouth

Permutation Patterns 2025
St Andrews, Scotland

Two ways to write permutations

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- in *one-line notation*, e.g. $\pi = 3\,1\,7\,6\,5\,4\,2$,
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Example

$\mathcal{S}_3^o = \{(1,2,3),(1,3,2),(1)(2)(3)\}$,
$\mathcal{S}_3^e = \{(1,2)(3),(1,3)(2),(2,3)(1)\}$.

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$$|\mathcal{S}_n^o| = |\mathcal{S}_n^e| = \begin{cases} (n-1)!!^2 = (n-1)^2(n-3)^2\dots 1^2 & \text{if } n \text{ is even,} \\ n\,(n-2)!!^2 = n(n-2)^2(n-4)^2\dots 2^2 & \text{if } n \text{ is odd.} \end{cases}$$

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1. Write each cycle of $\pi \in \mathcal{S}_n^o$ starting with its largest element, and order the cycles by increasing first element, e.g. $\pi = (4)(5,1,3)(7,2,6)(8)$.

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1. Write each cycle of $\pi \in \mathcal{S}_n^o$ starting with its largest element, and order the cycles by increasing first element, e.g. $\pi = (4)(5,1,3)(7,2,6)(8)$.
2. Move the last element of the 1st cycle to the end of the 2nd cycle, the last element of the 3rd cycle to the end of the 4th cycle, etc., e.g.

$$(4)(5,1,3)(7,2,6)(8) \mapsto (5,1,3,4)(7,2)(8,6) \in \mathcal{S}_n^e.$$

A surprising refinement

Definition

For $\pi = \pi_1\pi_2\ldots\pi_n \in \mathcal{S}_n$,

- its descent set is $\mathrm{Des}(\pi) = \{i : \pi_i > \pi_{i+1}\}$,
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Theorem (Adin, Hegedűs, Roichman '25)

For any n and any subset $J \subseteq [n-1]$,

$$|\{\pi \in \mathcal{S}_n^o : \mathrm{Asc}(\pi) = J\}| = |\{\pi \in \mathcal{S}_n^e : \mathrm{Des}(\pi) = J\}|.$$

A surprising refinement

$$|\{\pi \in \mathcal{S}_n^o : \mathrm{Asc}(\pi) = J\}| = |\{\pi' \in \mathcal{S}_n^e : \mathrm{Des}(\pi') = J\}|$$

Example ($n = 4$)

$\pi \in \mathcal{S}_4^o$	$\mathrm{Asc}(\pi) = \mathrm{Des}(\pi')$	$\pi' \in \mathcal{S}_4^e$
$(1,2,4)(3) = 2431$	$\{1\}$	$(1,4,3,2) = 4123$
$(1,4,2)(3) = 4132$	$\{2\}$	$(1,2,4,3) = 2413$
$(1,3,4)(2) = 3241$	$\{2\}$	$(1,3)(2,4) = 3412$
$(1,4,3)(2) = 4213$	$\{3\}$	$(1,2,3,4) = 2341$
$(2,3,4)(1) = 1342$	$\{1,2\}$	$(1,4,2,3) = 4312$
$(2,4,3)(1) = 1423$	$\{1,3\}$	$(1,3,4,2) = 3142$
$(1,2,3)(4) = 2314$	$\{1,3\}$	$(1,2)(3,4) = 2143$
$(1,3,2)(4) = 3124$	$\{2,3\}$	$(1,3,2,4) = 3421$
$(1)(2)(3)(4) = 1234$	$\{1,2,3\}$	$(1,4)(2,3) = 4321$

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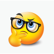
Unfortunately, Bóna's bijection $\mathcal{S}_n^o \to \mathcal{S}_n^e$ does not behave well with respect to Asc and Des.

For any n and any subset $S \subseteq [n-1]$, we will construct an explicit bijection

$$\{\pi \in \mathcal{S}_n^o : \mathrm{Asc}(\pi) \subseteq S\} \leftrightsquigarrow \{\pi \in \mathcal{S}_n^e : \mathrm{Des}(\pi) \subseteq S\}.$$

Structure of the bijection



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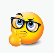

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Multisets of odd, distinct necklaces — Multisets of even necklaces

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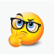

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$\downarrow$		$\downarrow$
Multisets of odd, distinct necklaces		Multisets of even necklaces
$\|\|\|$		$\|\|\|$
Words whose Lyndon factors are odd and distinct	$\longleftrightarrow$	Words whose Lyndon factors are even

Multisets of necklaces

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For a set $S = \{s_1, s_2, \dots, s_k\} \subseteq [n-1]$ with $s_1 < \dots < s_k$, let $\alpha = (s_1, s_2 - s_1, \dots, n - s_k)$ and $\mathrm{wt}(S) = \mathtt{a}^{\alpha_1}\mathtt{b}^{\alpha_2}\mathtt{c}^{\alpha_3}\dots$.

Example: If $S = \{2, 3\} \subseteq [5]$, then $\alpha = (2, 1, 3)$ and $\mathrm{wt}(S) = \mathtt{a}^2\mathtt{b}\mathtt{c}^3$.

From permutations to multisets of necklaces: $\mathrm{Des} \subseteq S$

In 1993, Gessel and Reutenauer described a bijection

$$\Phi_S : \{\pi \in \mathcal{S}_n : \mathrm{Des}(\pi) \subseteq S\} \to \{M \in \mathcal{M}_n : \mathrm{wt}(M) = \mathrm{wt}(S)\}$$

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Let $n = 8$ and $S = \{4, 7\}$, so $\mathrm{wt}(S) = \mathtt{a}^4\mathtt{b}^3\mathtt{c}$.

Take $\pi = 4567{\cdot}238{\cdot}1 \in \mathcal{S}_n$, which has $\mathrm{Des}(\pi) = S$.

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The multiset of necklaces $(\mathtt{a}, \mathtt{b})(\mathtt{a}, \mathtt{b})(\mathtt{a}, \mathtt{a}, \mathtt{b}, \mathtt{c})$ gives periodic sequences $(\mathtt{abab}\ldots, \mathtt{baba}\ldots)(\mathtt{abab}\ldots, \mathtt{baba}\ldots)(\mathtt{aabc}\ldots, \mathtt{abca}\ldots, \mathtt{bcaa}\ldots, \mathtt{caab}\ldots)$.

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To recover π from the multiset of necklaces:

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- Order these sequences lexicographically (breaking ties consistently).

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We get $\pi = (3,6)(2,5)(1,4,7,8)$.

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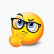

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However, those of *odd length* are primitive 😀

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From permutations to multisets of necklaces: $\mathrm{Asc} \subseteq S$

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However, those of *odd length* are primitive 😀 and **distinct**! 😲

$\mathcal{M}_n^o =$ multisets of distinct necklaces of odd length.

We get a bijection

$$\Xi_S : \{\pi \in \mathcal{S}_n^o : \mathrm{Asc}(\pi) \subseteq S\} \to \{M \in \mathcal{M}_n^o : \mathrm{wt}(M) = \mathrm{wt}(S)\}.$$

The bijections so far

$\mathcal{M}_n^e =$ multisets of necklaces of even length,
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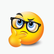

$$\{\pi \in \mathcal{S}_n^o : \mathrm{Asc}(\pi) \subseteq S\} \quad \leftrightsquigarrow \quad \{\pi \in \mathcal{S}_n^e : \mathrm{Des}(\pi) \subseteq S\}$$

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We will interpret multisets of necklaces as words.

Lyndon words

A Lyndon word is a primitive word that is lexicographically smaller than all of its cyclic rotations. Denote the set of Lyndon words by $\mathcal{L} \subseteq \mathcal{W}$.

Example: $\texttt{aabab} \in \mathcal{L}$, but $\texttt{ababa} \notin \mathcal{L}$.

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We identify necklaces with Lyndon words, e.g., $(\mathtt{a}, \mathtt{b}, \mathtt{a}, \mathtt{b}, \mathtt{a}) \leftrightarrow \mathtt{aabab}$.

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Theorem (Lyndon '58)

Every $w \in \mathcal{W}$ has a unique Lyndon factorization $w = \ell_1|\ell_2|\dots|\ell_m$ where $\ell_i \in \mathcal{L}$ for all i, and $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_m$ lexicographically.

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We identify multisets of necklaces with words.

The bijections so far

Define the following sets of length-n words:

$\mathcal{W}_n^e =$ words all of whose Lyndon factors have even length, except possibly for one factor which has length one.

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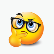

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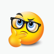

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\{w \in \mathcal{W}_n^o : \mathrm{wt}(w) = \mathrm{wt}(S)\} & \stackrel{?}{\longleftrightarrow} & \{w \in \mathcal{W}_n^e : \mathrm{wt}(w) = \mathrm{wt}(S)\}
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We want a weight-preserving bijection between $\mathcal{W}_n^o$ and $\mathcal{W}_n^e$.

A weight preserving bijection $\Psi : \mathcal{W}_n^o \to \mathcal{W}_n^e$

Given $w \in \mathcal{W}_n^o$ (suppose n is even), initially set $O = w$ and $E = -$.

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- Update (O, E) to

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Set $\Psi(w) = E$.

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- Say that o_m is *splittable* if $s < o_{m-1}$ (where $o_{m-1} := \infty$ if $m = 1$).
- $$(O, E) \mapsto \begin{cases} (o_1 o_2 \dots o_{m-1} r,\ sE) & \text{if } o_m \text{ splittable and } |s| \text{ even,} \\ (o_1 o_2 \dots o_{m-1} s,\ rE) & \text{if } o_m \text{ splittable and } |r| \text{ even,} \\ (o_1 o_2 \dots o_{m-2},\ o_m o_{m-1} E) & \text{if } o_m \text{ not splittable.} \end{cases}$$

Example

	O	E
$w =$	d|adcdebccc	–
	d|adcde	bccc
	d|c¦de	adbccc

A weight preserving bijection $\Psi : \mathcal{W}_n^o \to \mathcal{W}_n^e$

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A weight preserving bijection $\Psi : \mathcal{W}_n^o \to \mathcal{W}_n^e$

Theorem

The map $\Psi : \mathcal{W}_n^o \to \mathcal{W}_n^e$ *is a weight-preserving bijection.*

A weight preserving bijection $\Psi : \mathcal{W}_n^o \to \mathcal{W}_n^e$

Theorem

The map $\Psi : \mathcal{W}_n^o \to \mathcal{W}_n^e$ is a weight-preserving bijection.

Composing the three maps, we obtain the desired bijection:

$$\begin{array}{ccc}
\{\pi \in \mathcal{S}_n^o : \mathrm{Asc}(\pi) \subseteq S\} & \overset{\Phi_S^{-1} \circ \Psi \circ \Xi_S}{\rightsquigarrow} & \{\pi \in \mathcal{S}_n^e : \mathrm{Des}(\pi) \subseteq S\} \\
\Xi_S \downarrow & & \downarrow \Phi_S \\
\{M \in \mathcal{M}_n^o : \mathrm{wt}(M) = \mathrm{wt}(S)\} & & \{M \in \mathcal{M}_n^e : \mathrm{wt}(M) = \mathrm{wt}(S)\} \\
\wr\wr\wr & & \wr\wr\wr \\
\{w \in \mathcal{W}_n^o : \mathrm{wt}(w) = \mathrm{wt}(S)\} & \overset{\Psi}{\longrightarrow} & \{w \in \mathcal{W}_n^e : \mathrm{wt}(w) = \mathrm{wt}(S)\}
\end{array}$$

Bonus: a short generating function proof

$$|\{w \in \mathcal{W}_n^o : \mathrm{wt}(w) = x_1^{\alpha_1} x_2^{\alpha_2} \dots\}| = |\{w \in \mathcal{W}_n^e : \mathrm{wt}(w) = x_1^{\alpha_1} x_2^{\alpha_2} \dots\}|$$

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Proof:

- Write $\mathcal{L} = \mathcal{L}^o \sqcup \mathcal{L}^e$, separating Lyndon words of odd and even length.

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- GF for $\mathcal{W}^o = \bigcup_{n \geq 0} \mathcal{W}_n^o$ (words with odd and distinct Lyndon factors):

$$\sum_{w \in \mathcal{W}^o} \mathrm{wt}(w) = \prod_{\ell \in \mathcal{L}^o} (1 + \mathrm{wt}(\ell)) .$$

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- GF for $\mathcal{W}^e = \bigcup_{n\geq 0} \mathcal{W}_n^e$ (even Lyndon factors, except one of length 1):
$$\sum_{w\in\mathcal{W}^e} \mathrm{wt}(w) = (1 + x_1 + x_2 + \dots) \prod_{\ell\in\mathcal{L}^e} \frac{1}{1 - \mathrm{wt}(\ell)}.$$

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- Thus, what we want to prove is

$$\prod_{\ell \in \mathcal{L}^o} (1 + \mathrm{wt}(\ell)) = (1 + x_1 + x_2 + \dots) \prod_{\ell \in \mathcal{L}^e} \frac{1}{1 - \mathrm{wt}(\ell)},$$

Bonus: a short proof using generating functions

or equivalently,

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- Substituting $x_i \mapsto -x_i$ for all i, this is equivalent to

$$\prod_{\ell \in \mathcal{L}^o} (1 - \mathrm{wt}(\ell)) \prod_{\ell \in \mathcal{L}^e} (1 - \mathrm{wt}(\ell)) = 1 - x_1 - x_2 - \dots,$$

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- But this holds because every word has a unique Lyndon factorization!

□