Pattern avoidance and Schur-positivity in restricted-growth words of type B

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Some history

Klazar (1996), Sagan (2006):

- Two different approaches to pattern-avoidance in set partitions of type A:
  1. Following Klazar, based on avoiding some order of elements in the blocks of the set partitions (where the blocks are ordered increasingly with respect to their minimal elements).
  2. Based on their restricted-growth words representation.

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- Pattern-avoidance in the set partitions sense implies pattern-avoidance in the RG-words representation sense, but not the other way around.
- Characterization of all the pattern-avoidance classes of one pattern of size 3.

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- Characterization of all the pattern-avoidance classes of one pattern of size 3.

Goyt (2008): Pattern-avoidance classes of some families of set partitions of type A.

Some history (cont.)

Bloom and Elizalde (2013): Perfect matchings and partitions that avoid patterns of length 3, based on a different notion of pattern-avoidance for set partitions that involve arc-diagrams.

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Mansour et al. (2008/2012): Full characterization of all classes of set partitions that avoid a pattern of lengths 3,4,5 or avoid two such patterns, with respect to their RG-word representation.

Set partitions of type B (Dolgachev-Lunts & Reiner)

Definition

Denote: $[\pm n] := \{\pm 1, \dots, \pm n\}$.

A **set partition of $[n]$ of type B** or a **signed set partition** is a set partition of the set $[\pm n]$ satisfying the following conditions:

- If B appears as a block, then $-B$ (which is obtained from B by negating all its elements) also appears in that partition.
- There exists at most one block satisfying $-B = B$. This block is called the **zero block**, denoted by B_0.
  If it exists, $B = \{\pm i \mid i \in C\} \subseteq [\pm n]$ for some $C \subseteq [n]$.

Example

$$\left\{ \begin{array}{c} B_0 = \{1, -1, 4, -4\},\ B = \{2, 3, -5\}, -B = \{-2, -3, 5\}, \\ B' = \{6\}, -B' = \{-6\} \end{array} \right\}.$$

Set partitions of type B (Dolgachev-Lunts & Reiner) (cont.)

The **representative block** for the pair of blocks $B, -B$ is the one containing the minimal **positive** number appearing in $B \cup -B$.

Example

The block $\{2, 3, -5\}$ represents the pair of blocks:

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The **standard presentation** of a set partition of type B is given by writing first the zero block if it exists, and then the non-zero representative blocks in such a way that the sequence of absolute values of the minimal elements of the blocks is increasing.

Example

$$\{B_0 = \{1, -1, 4, -4\}, B_1 = \{\mathbf{2}, 3, -5\}, B_2 = \{\mathbf{6}\}\}.$$

Definition

Let $\Sigma^B = \{0, \pm 1, \pm 2, \dots, \pm n\}$ and define the following order on Σ^B: $0 \prec -1 \prec 1 \prec -2 \prec 2 \prec \cdots \prec -n \prec n$.

A **restricted-growth (RG-)word of type B of the second kind of length n** is a word $\omega = \omega_1 \cdots \omega_n$ in the alphabet Σ^B, which satisfies:

(1) $\omega_1 = 0$ or $\omega_1 = 1$.

(2) For each $2 \leq t \leq n$: $\omega_t \preceq \max\{|\omega_1|, \dots, |\omega_{t-1}|\} + 1$, with respect to the order defined above. In the case that $|\omega_t| = \max\{|\omega_1|, \dots, |\omega_{t-1}|\} + 1$, we demand: $\omega_t > 0$.

Denote by $R^B(n,k)$ the set of RG-words of type B of length n whose maximal element is k. Define: $\mathcal{R}_n^B = \bigcup\limits_k R^B(n,k)$.

Given a set partition of $[n]$ of type B, written in its standard presentation, $P = \{B_0, B_1, \ldots, B_k\}$, we associate to it an RG-word $\omega = \omega_1 \cdots \omega_n$ of type B:

- For each $1 \leq j \leq n$, ω_j is the number of the representative block where j or $-j$ appears.
- If j appears in the representative block, then ω_j is the number of the block containing j; otherwise, ω_j is the number of this block, with a negative sign.

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Example

Given a set partition of $[7]$ of type B:

$$P = \{B_0 = \{2, 5, -2, -5\}, B_1 = \{\mathbf{1}, -7\}, B_2 = \{\mathbf{3}, -4, 6\}\}.$$

Its associated RG-word of type B is:

$$\omega = 102(-2)02(-1).$$

Definition

Let $\omega = \omega_1 \cdots \omega_n \in \mathcal{R}_n^B$ and let $\tau = \tau_1 \cdots \tau_k \in \Sigma_B^*$ for some k, such that $\{|\tau_1|, \ldots, |\tau_k|\} = \{0, 1, \ldots, d\}$ for some $d \leq n$.

We say that ω **contains** τ, if there are k indices,

$$1 \leq i_1 < i_2 < \cdots < i_k \leq n$$

such that $\omega_{i_a} @ \omega_{i_b}$ if and only if $\tau_a @ \tau_b$, for all $1 \leq i < j \leq k$ and $@ \in \{\prec, =, \succ\}$.

τ is usually called a **signed pattern**. If ω does not contain τ, we say that ω **avoids** τ. For an arbitrary finite collection of patterns P, we say that ω **avoids** P if ω avoids each $\tau \in P$.

Example

The RG-word 00**1(-1)(-1)**21 contains 100, but avoids 210.

The generating tree $\mathcal{T}(P)$ (following Mansour *et al.*)

Notation: $\mathcal{R}_n^B(\tau)$ is the set of all τ-avoiding RG-words of signed set partitions in $\mathcal{R}_n^B$.
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Starting with the root ϵ (which occupies level 0), the children of the node $\omega_1 \cdots \omega_{n-1} \in \mathcal{R}_{n-1}^B(P)$ in level $n-1$ are all the elements $\omega_1 \cdots \omega_{n-1}\omega_n \in \mathcal{R}_n^B(P)$.

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For a given set of patterns P and an RG-word ω, let $\mathcal{T}(P;\omega)$ denote the subtree in $\mathcal{T}(P)$ whose root is ω.

We define an equivalence relation on the set of nodes of $\mathcal{T}(P)$: $\omega \sim \omega'$ if the subtrees $\mathcal{T}(P;\omega), \mathcal{T}(P;\omega')$ are isomorphic in the sense of ordered trees.
For every equivalence class, we choose a unique representative.

Let $\mathcal{T}[P]$ be the same tree as $\mathcal{T}(P)$, where we change every label of each node to be its unique representative of corresponding class.

The main procedure for computing the generating function

(1) After setting the first succession rule $\epsilon \rightsquigarrow 0, 1$ (that describes the children of the empty word as 0 and 1), we make an educated guess for the succession rules of $\mathcal{T}[P]$ by computing the first m levels of $\mathcal{T}[P]$. Then, we prove that this set is indeed the entire set of succession rules for $\mathcal{T}[P]$.

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(2) For each succession rule of the type $v \rightsquigarrow v^{(1)}, v^{(2)}, \ldots, v^{(\ell)}$, define $A_v(x)$ as the generating function for the number of nodes at level n in the subtree $\mathcal{T}(P; v)$ of $\mathcal{T}[P]$, where the root of this subtree is the vertex v. Then each succession rule of the type $v \rightsquigarrow v^{(1)}, v^{(2)}, \ldots, v^{(\ell)}$ can be translated into the equation:

$$A_v(x) = 1 + x \sum_{j=1}^{\ell} A_{v^{(j)}}(x).$$

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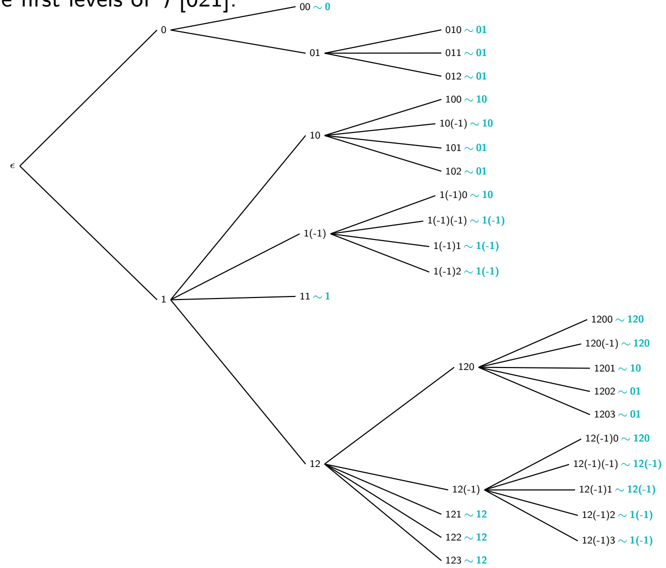
(3) Solve the obtained system of equations for finding $A_\epsilon(x)$, using different types of techniques, such as guessing the solution and then proving it or the kernel method.

An example of the procedure using generating trees

We construct the first levels of $\mathcal{T}[021]$:



Hence, we have the following succession rules:

$$\begin{aligned}
\epsilon &\rightsquigarrow 0, 1, \\
0 &\rightsquigarrow 0, 01, \\
1 &\rightsquigarrow 1(-1), 10, 1, 12, \\
01 &\rightsquigarrow 01, 01, 01, \\
1(-1) &\rightsquigarrow 10, 1(-1), 1(-1), 1(-1), \\
10 &\rightsquigarrow 10, 10, 01, 01, \\
12 &\rightsquigarrow 12(-1), 120, 12, 12, 12, \\
12(-1) &\rightsquigarrow 120, 12(-1), 12(-1), 1(-1), 1(-1), \\
120 &\rightsquigarrow 120, 120, 10, 01, 01.
\end{aligned}$$

An example of the procedure using generating trees (cont.)

Next, we translate the succession rules into functional equations:

$$\begin{aligned}
A_\epsilon(x) &= 1 + xA_0(x) + xA_1(x),\\
A_0(x) &= 1 + xA_0(x) + xA_{01}(x),\\
A_1(x) &= 1 + xA_{1(-1)} + xA_{10}(x) + xA_1(x) + xA_{12}(x),\\
A_{01}(x) &= 1 + 3xA_{01}(x),\\
A_{1(-1)}(x) &= 1 + 3xA_{1(-1)}(x) + xA_{10}(x),\\
A_{10}(x) &= 1 + 2xA_{10}(x) + 2xA_{01}(x),\\
A_{12}(x) &= 1 + xA_{12(-1)}(x) + xA_{120}(x) + 3xA_{12}(x),\\
A_{12(-1)}(x) &= 1 + 2xA_{12(-1)}(x) + xA_{120}(x) + 2xA_{1(-1)}(x),\\
A_{120}(x) &= 1 + 2xA_{120}(x) + xA_{10}(x) + 2xA_{01}(x).
\end{aligned}$$

Solving for $A_\epsilon(x)$:

$$F_{021}(x) = A_\epsilon(x) = \frac{5x^7 - 168x^6 + 421x^5 - 449x^4 + 256x^3 - 82x^2 + 14x - 1}{(3x-1)^3(2x-1)^3(x-1)}.$$

Theorem (BGMS 2025)

- $\sum_{n\geq 0} \left|\mathcal{R}_n^B(00)\right| \frac{x^n}{n!} = (1+x)e^{x+\frac{x^2}{2}}$,
- $\sum_{n\geq 0} \left|\mathcal{R}_n^B(01)\right| x^n = \frac{1-x+x^2}{(1-x)^3}$,
- $\sum_{n\geq 0} \left|\mathcal{R}_n^B(10)\right| x^n = \frac{1}{1-2x}$.

Avoiding patterns of length 3 - Enumerative results (BGMS 2025)

τ	$\{\lvert\mathcal{R}_n^B(\tau)\rvert\}_{n=1}^9$	Enum.
000	2, 6, 22, 98, 486, 2692, 16346, 107382, 756748	
001, 010	2, 6, 22, 87, 357, 1517, 6677, 30407, 143027	
011	2, 6, 22, 87, 361, 1554, 6907, 31609, 148664	e✓
012	2, 6, 22, 84, 315, 1138, 3941, 13093, 41857	o✓
021	2, 6, 22, 86, 339, 1322, 5069, 19084, 70583	o✓
100	2, 6, 22, 92, 432, 2224, 12392, 74064, 470944	e✓
101, 110	2, 6, 22, 91, 412, 2002, 10306, 55709, 314146	
102, 120	2, 6, 22, 85, 330, 1276, 4916, 18901, 72602	o✓
201, 210	2, 6, 23, 100, 467, 2285, 11559, 59960, 317201	o✓

We denote $o\checkmark$ ($e\checkmark$) where we have computed the ordinary (exponential) generating function.

None of the sequences appears in the OEIS.

Avoiding patterns of length 2 - Schur-positivity

Let $\mathbf{x} = \{x_1, x_2, \ldots\}$ be a countably infinite set of commuting variables and let $\mathbb{Q}[[\mathbf{x}]]$ be the algebra of formal power series over $\mathbb{Q}$.

A power series $f \in \mathbb{Q}[[\mathbf{x}]]$ is called **symmetric** if it has a bounded degree and it is invariant under **any** permutation of variables.

Denote by Sym_n the vector space of symmetric functions, homogeneous of degree n. Each basis for Sym_n is indexed by partitions λ of n, or equivalently, by Young diagrams with n boxes.

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Definition

A **quasi-symmetric function** is a formal power series g of bounded degree satisfying that any two of its monomials $x_{i_1}^{n_1} \dots x_{i_k}^{n_k}$ (where $i_1 < \cdots < i_k$) and $x_{j_1}^{n_1} \dots x_{j_k}^{n_k}$ (where $j_1 < \cdots < j_k$) have the same coefficient in g.

Avoiding patterns of length 2 - Schur-positivity (cont.)

The most important basis for Sym_n is the set of Schur functions

$$\{s_\lambda \mid \lambda \vdash n\}.$$

Definition

A symmetric function is called **Schur-positive** if all coefficients in its expansion in the basis of Schur functions are nonnegative.

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Definition

For each subset $D \subseteq [n-1]$, define the **fundamental quasi-symmetric function**: $F_{n,D}(\mathbf{x}) := \sum_{\substack{i_1 \le i_2 \le \ldots \le i_n \\ i_j < i_{j+1} \text{ if } j \in D}} x_{i_1} x_{i_2} \cdots x_{i_n}.$

The **fundamental basis** for the vector space of homogeneous quasi-symmetric functions of degree n over $\mathbb{Q}$ consists of the fundamental quasi-symmetric functions.

Avoiding patterns of length 2 - Schur-positivity (cont.)

Let $\mathcal{B}$ be a (multi-)set of combinatorial objects, equipped with a **descent map** $\mathrm{Des} : \mathcal{B} \to P([n-1])$, which associates to each element $b \in \mathcal{B}$ a subset $\mathrm{Des}(b) \subseteq [n-1]$. Define the quasi-symmetric function:

$$\mathcal{Q}_n(\mathcal{B}) := \sum_{b \in \mathcal{B}} m(b, \mathcal{B}) F_{n,\mathrm{Des}(b)},$$

where $m(b, \mathcal{B})$ is the multiplicity of the element $b \in \mathcal{B}$.

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The **descent set of a standard Young tableau T** is:

$$\mathrm{Des}(T) = \{i \mid i+1 \text{ is in a lower row than } i \text{ in } T\}.$$

Example

Given $T = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & 7 \\ \hline 6 \\ \cline{1-1}\end{array}$, we have: $\mathrm{Des}(T) = \{1, 3, 5\}$.

Avoiding patterns of length 2 - Schur-positivity (cont.)

Theorem (Gessel)

For every partition $\lambda \vdash n$, $\mathcal{Q}_n(\mathrm{SYT}(\lambda)) = s_\lambda$.

Hence, proving Schur-positivity of a set E with respect to some descent function, amounts to defining a descent-preserving bijection $\varphi : E \to \bigcup_{\lambda \vdash n} \mathrm{SYT}(\lambda)$, such that for each $\lambda \vdash n$ and $S, T \in \mathrm{SYT}(\lambda)$, one has $|\varphi^{-1}(S)| = |\varphi^{-1}(T)|$.

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Definition

Given a set of pattern RG-words P, let the **pattern quasi-symmetric function** $\mathcal{Q}_n(P) = \sum_{\omega \in R_n(P)} F_{n,\mathrm{Des}(\omega)}$.

Question

- *When is $\mathcal{Q}_n(P)$* **symmetric** *for all n?*
- *In that case, when is $\mathcal{Q}_n(P)$* **Schur-positive** *for all n?*

Avoiding patterns of length 2 - Schur-positivity (cont.)

Theorem (BGMS 2025)

$\mathcal{Q}_n(\mathcal{R}_n^B(01))$ is **Schur-positive** *with respect to ordinary descents of RG-words. Explicitly:* $\mathcal{Q}_n(\mathcal{R}_n^B(01)) = 2s_{(n)} + 2s_{(n-1,1)} + s_{(n-2,1,1)}$.

Example (Schur-positivity of $\mathcal{R}_4^B(01)$)

Signed partition	Associated RG-word	Des	SYT
$\{\{\pm1, \pm2, \pm3, \pm4\}\}$	0000	$\emptyset$	1 2 3 4
$\{\{1, 2, 3, 4\}\}$	1111	$\emptyset$	1 2 3 4
$\{\{1, 2, 3, -4\}\}$	$111(-1)$	$\{3\}$	1 2 3 4
$\{\{\pm4\}, \{1, 2, 3\}\}$	1110	$\{3\}$	1 2 3 4
$\{\{1, 2, -3, -4\}\}$	$11(-1)(-1)$	$\{2\}$	1 2 4 3
$\{\{\pm3, \pm4\}, \{1, 2\}\}$	1100	$\{2\}$	1 2 4 3
$\{\{1, -2, -3, -4\}\}$	$1(-1)(-1)(-1)$	$\{1\}$	1 3 4 2
$\{\{\pm2, \pm3, \pm4\}, \{1\}\}$	1000	$\{1\}$	1 3 4 2
$\{\{\pm4\}, \{1, 2, -3\}\}$	$11(-1)0$	$\{2, 3\}$	1 2 3 4
$\{\{\pm4\}, \{1, -2, -3\}\}$	$1(-1)(-1)0$	$\{1, 3\}$	1 3 2 4
$\{\{\pm3, \pm4\}, \{1, -2\}\}$	$1(-1)00$	$\{1, 2\}$	1 4 2 3

Thank you for your attention!!