Positional statistics for separable permutations

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Joint work with O. Lopez and M. Weiner

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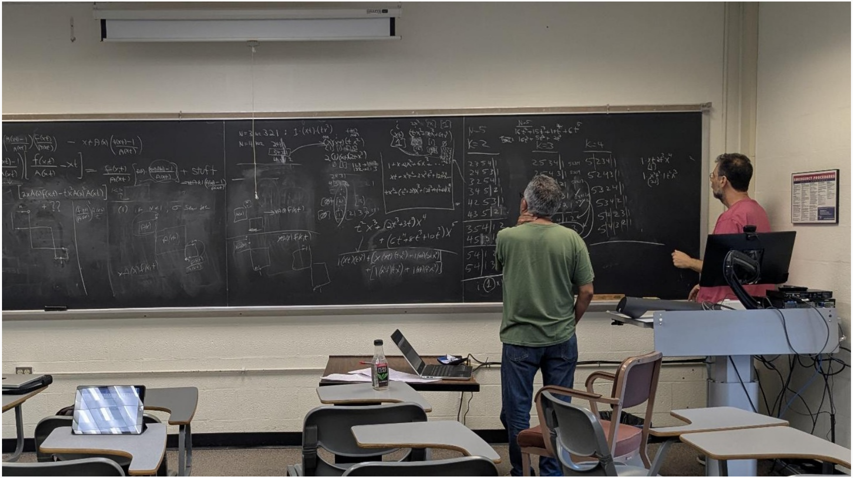
Joint work (in progress) with O. Lopez and M. Weiner

Working with MikeGPT



pc: Diego G.

Notation

For $a, k \geq 1$, we let

$$\mathcal{S}_{n,k}^{a \prec n}(P)$$

denote the set of $\sigma \in \mathcal{S}_n(P)$ such that:

- $\sigma^{-1}(n) - \sigma^{-1}(a) = k$,
- $\sigma^{-1}(b) - \sigma^{-1}(n) > 0$ for every $b \in \{1, \ldots, a-1\}$.

Motivating example: $\mathcal{S}_{n,k}^{1\prec n}(1324)$

For $n \geq 2$, there is a bijection between $\mathcal{S}_{n,1}^{1\prec n}(1324)$ and the set of 1324-avoiding dominoes* with $n-2$ points.

*D. Bevan, R. Brignall, A. Elvey Price, J. Pantone, A structural characterisation of Av(1324) and new bounds on its growth rate, *European J. Combin.* **88** (2020)

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Generating function:

$$f(x) = x + 2x^2 + 6x^3 + 22x^4 + 91x^5 + 408x^6 + \cdots \quad \text{(A000139)}$$

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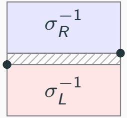
Bijection:

$$\sigma = \sigma_L \, 1 n \, \sigma_R \quad \mapsto$$



where σ_L and σ_R avoid 132 and 213, respectively.

Main results for 1324

Let

$$T_{a,k}(x) = \sum_{n=k+1}^{\infty} |\mathcal{S}_{n,k}^{a\prec n}(1324)| x^n \quad \text{and} \quad g_a(x,t) = \sum_{k=1}^{\infty} t^k T_{a,k}(x),$$

- $T_{1,k}(x) = xf(x)^k$ and $g_1(x,t) = \dfrac{xtf(x)}{1-tf(x)}$.
- $T_{2,1}(x) = \frac{1}{2}x^2 \frac{d}{dx} T_{1,1}(x)$ and

$$T_{2,k}(x) = x^2 f(x)^k + kf(x)^{k-1}\left(T_{2,1}(x) - x^2 f(x)\right).$$

- $g_2(x,t) = \frac{1}{2}\left(x^2 \frac{\partial g_1}{\partial x}(x,t) - g_1(x,t)^2\right)$.

Conjecture

For $k \geq a$, we have the equivalent formulas:

(i) $\sum_{j=0}^{k} (-1)^j \binom{k}{j} f(x)^j T_{a,k-j} = 0.$

(ii) $T_{a,k}(x) = \sum_{j=0}^{a-1} \binom{k}{j} f(x)^{k-j} \sum_{i=0}^{j} (-1)^i \binom{j}{i} f(x)^i T_{a,j-i}(x).$

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Why is this relevant? If $G(x) = \sum_{n=1}^{\infty} |\mathcal{S}_n(1324)| x^n$, then

$$G(x) = \frac{1}{1-x}\left(x + \sum_{a=1}^{\infty} g_a(x,1)\right).$$

Proof of concept: Separable permutations

Let

$$S(x) = 1 + x + 2x^2 + 6x^3 + 22x^4 + 90x^5 + \cdots$$

be the generating function for $|\mathcal{S}_n(2413, 3142)|$, and let

$$\mathcal{S}_n^{\ell \mapsto 1} = \{\sigma \in \mathcal{S}_n(2413, 3142) : \sigma(\ell) = 1\}.$$

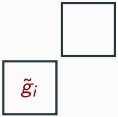
Proposition

The function $g(x,u) = \sum_{n=1}^{\infty} \sum_{\ell=1}^{n} |\mathcal{S}_n^{\ell \mapsto 1}|\, u^{\ell} x^n$ satisfies

$$g(x,u) = \frac{xuS(x)S(xu)}{S(x) + S(xu) - S(x)S(xu)}.$$

Proof

1. Indecomposable/decomposable
   - $g(x, u) = (xu + g_i(x, u))S(x)$
   - $g_d(x, u) = (xu + g_i(x, u))(S(x) - 1)$



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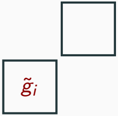

2. Involution from reverse map
   - $g(x,u) = ug(xu, \frac{1}{u})$
   - $g_d(x,u) = ug_i(xu, \frac{1}{u})$ (separable)

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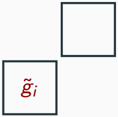

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   - $g(x,u) = ug(xu, \frac{1}{u})$
   - $g_d(x,u) = ug_i(xu, \frac{1}{u})$ (separable)
3. Some basic algebra...

$$\begin{aligned} g(x,u) &= ug(xu, \tfrac{1}{u}) = (xu + ug_i(xu, \tfrac{1}{u}))S(xu) \\ &= (xu + g_d(x,u))S(xu) \\ &= (xu + (xu + g_i(x,u))(S(x) - 1))S(xu) \end{aligned}$$

leads to

$$g_i(x,u) = \frac{xuS(x)(S(xu) - 1)}{S(x) + S(xu) - S(x)S(xu)}.$$

Enumeration of separable permutations

ELSEVIER

Discrete Mathematics 146 (1995) 247–262

DISCRETE MATHEMATICS

Generating trees and the Catalan and Schröder numbers

Julian West

Department of Computer Science, Bordeaux University, Bordeaux, France

Received 25 August 1991; revised 28 January 1994

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⋮

The data we have examined support the following conjecture.

Conjecture 4.4. Among all the permutations of $S_n(3142, 2413)$, take those in which 1 appears in position j. For each of these, count 1 less than the number of active sites (with respect to 3142 and 2413). Then the total is $s_{j-1} \cdot s_{n-j}$.

Enumeration of separable permutations

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DISCRETE MATHEMATICS

Forbidden subsequences

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Received 26 November 1991; revised 17 November 1992

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a special case for $n=4$. The proof presented by West in [10] gives a description of the tree $T(2413, 3142)$ and uses generating functions. In search of a better proof he made the following conjecture (4.2.1, p. 30): Among all the permutations of length n, take those in which 1 appears in position j. For each of these, count 1 less than the number of active sites (with respect to the forbidden patterns 3142 and 2413). Then the total is $s_j \cdot s_{n-j}$. If this conjecture were true, one could easily prove the Schröder result using the recurrence relationship for the s_n's. Unfortunately, it is not. Consider, for example,

Bonus

We just proved that

$$g(x, u) = \frac{xuS(x)S(xu)}{S(x) + S(xu) - S(x)S(xu)}.$$

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$$g(x,u) = \frac{xuS(x)S(xu)}{S(x) + S(xu) - S(x)S(xu)}.$$

If we let $u = 1$, then

$$S(x) - 1 = \frac{xS(x)S(x)}{S(x) + S(x) - S(x)S(x)} = \frac{xS(x)}{2 - S(x)},$$

which leads to $S(x) = \frac{1}{2}\big(3 - x - \sqrt{1 - 6x + x^2}\big)$.

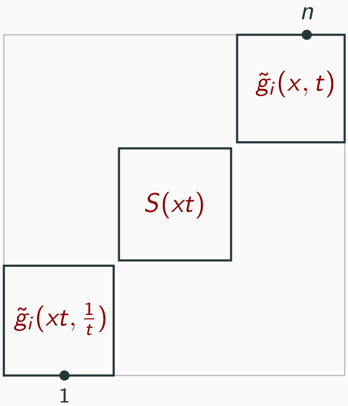
Enumeration by relative position

Theorem

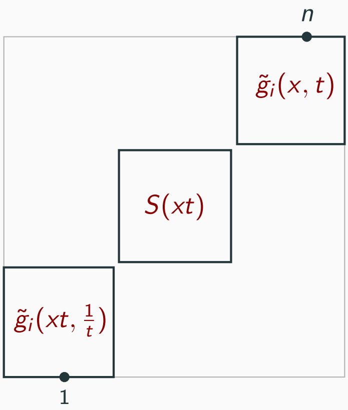
For $f(x,t) = \sum_{n,k\geq 1} |\mathcal{S}_{n,k}^{1\prec n}(2413, 3142)| t^k x^n$, we have

$$f(x,t) = \frac{x^2 t S(x) S(xt)^2}{\big(S(xt) + S(x) - S(xt)S(x)\big)^2}.$$

Sketch of proof



Sketch of proof

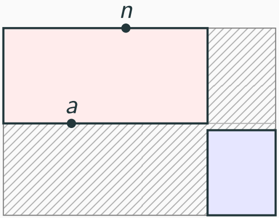


$$f(x,t) = \left(x + g_i(xt, \tfrac{1}{t})\right) S(xt) \left(xt + g_i(x,t)\right).$$

$$g_i(x,u) = \frac{xuS(x)(S(xu)-1)}{S(x)+S(xu)-S(x)S(xu)}$$

2413-property and corollary

For $a > 1$, any $\sigma \in \mathcal{S}_{n,k}^{a \prec n}(2413)$ must be of the form $\sigma = \pi \ominus \tau$, where $\pi \in \mathcal{S}_{m,k}^{1 \prec m}(2413)$ with $m = n - a + 1$, and $\tau \in \mathcal{S}_{a-1}(2413)$.



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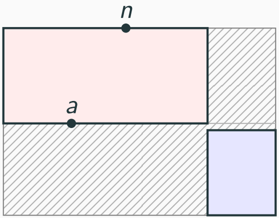

If $F(x,t,s) = \sum_{n,k,a \geq 1} |\mathcal{S}_{n,k}^{a \prec n}(2413, 3142)| t^k s^a x^n$, then

$$F(x,t,s) = f(x,t) \cdot sS(xs).$$

Note that 3142 is skew indecomposable!

Open problems

- Enumeration of $S_n^{a \prec n}(1324)$ for $a \geq 3$

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- ▷ Enumeration of 2413 by the position of the 1

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- Positional statistics for Baxter permutations

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- Enumeration of 2413 by the position of the 1
- Positional statistics for Baxter permutations
- Positional statistics for patterns of size 4
- Positional statistics for pair of patterns of size 4

Thank you!