Emerging consecutive pattern avoidance

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Definition

Let $\pi = a_1 \ldots a_n \in \mathcal{S}_n$, and $p \in \mathcal{S}_3$.

- We say that π contains a consecutive occurrence of the pattern p if there exists a subsequence of consecutive letters $a_i a_{i+1} a_{i+2}$ of π that is order-isomorphic to p.
- We say that π *avoids* the consecutive pattern p if it does not contain any consecutive occurrence of p.

Example: The permutation 6437215 contains

Multi-avoidance of length 3 consecutive patterns

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Example: The permutation $\underline{643}\overline{721}5$ contains

- 2 occurrences of 321

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Example: The permutation $6\underline{437}\overline{215}$ contains

- 2 occurrences of 321
- 2 occurrences of 213

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Example: The permutation $64\underline{372}15$ contains

- 2 occurrences of 321
- 2 occurrences of 213
- 1 occurrence of 231

3 bijections preserving the occurrences of consecutive patterns:

- The *reverse* of $\pi = a_1 \dots a_n$ is $R(\pi) = a_n \dots a_1$.
- The *complement* is $C(\pi) = (n+1-a_1)\dots(n+1-a_n)$.
- The *reverse-complement* $R \circ C$ is the composition of R and C.

Fact

For $T \in \{R, C, R \circ C\}$, $\pi \in \mathcal{S}_n$ and a consecutive pattern p, π has an occurrence of p if and only if $T(\pi)$ has an occurrence of $T(p)$.

Classes of multi-avoidance

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There are 18 classes of multi-avoidance of length 3 consecutive patterns under the action of $\{Id, R, C, R \circ C\}$.

Classes of multi-avoidance

Kitaev and Mansour gave the enumeration of all those 18 classes:

- Kitaev, *Multi-avoidance of generalised patterns* (2003).
- Kitaev and Mansour, *Simultaneous avoidance of generalized patterns* (2005).
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Question: among a class, what are the asymptotic frequencies of the allowed patterns?

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Question: among a class, what are the asymptotic frequencies of the allowed patterns?

Definition

Let $\mathcal{A}_n := \mathrm{Av}_n(p_1, \ldots, p_m)$. For a pattern $p \notin \{p_1, \ldots, p_m\}$, we denote by $\mathrm{p}_n^{\mathcal{A}}$ the total number of occurrences of p in $\mathcal{A}_n$. We define the *asymptotic popularity* of p in the class $\mathcal{A}$ by

$$\mathrm{pop}_{\mathcal{A}}(p) := \lim_{n\to\infty} \frac{\mathrm{p}_n^{\mathcal{A}}}{n|\mathcal{A}_n|}.$$

Avoiding 132 and 231

Lemma - *Kitaev (2003)*

A permutation $\pi \in \mathsf{Av}_n(132, 231)$ has the following form:

$$\pi = a_1 \dots a_k 1 b_1 \dots b_{n-k-1},$$

where $a_1 \dots a_k$ is a decreasing sequence, and $b_1 \dots b_{n-k-1}$ an increasing sequence.

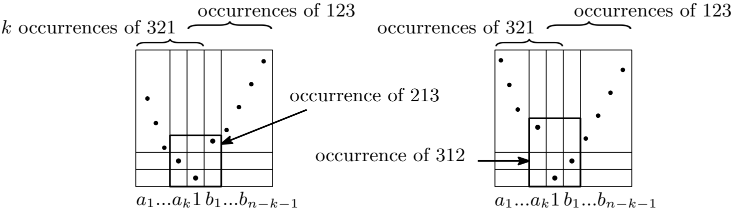

Figure: General structure of a permutation from $\mathsf{Av}_n(132, 231)$.

Class \ Pattern	123	132	213	231	312	321
1 (simple)			1/2	1/2		
2 (simple)				0		1
3 (simple)	1/2					1/2
4 (simple)			N/A		N/A	
5 (simple)	1			0		
6 (simple)				1/2	1/2	
7 (done in [⋆])				1/2	1/2	0
8 (simple)			0		0	1
9 (simple)	1/2				0	1/2
10 (open)			?	?		?
11			1/4	1/2	1/4	
12 (open)		?	?			?
13 (open)		?	?		?	?
14 (open)	?	?			?	?
15 (open)	?			?	?	?
16 (simple)		1/4	1/4	1/4	1/4	
17			1/4	1/2	1/4	0
18 (simple)	1/2		0		0	1/2

Figure: The asymptotic popularity of patterns among 18 avoidance classes.

[⋆] Baril, Burstein and Kirgizov, *Pattern statistics in faro words and permutations* (2021).

Avoiding 123,132 and 321

Theorem - *Kitaev (2003)*

$$|\mathrm{Av}_n(123, 132, 321)| = (n-1)!! + (n-2)!!,$$

where $n!!$ is defined by $0!! = 1$, and for $n \geq 1$

$$n!! = \begin{cases} n \cdot (n-2) \ldots 3 \cdot 1 & \text{if } n \text{ is odd,} \\ n \cdot (n-2) \ldots 4 \cdot 2 & \text{if } n \text{ is even.} \end{cases}$$

Avoiding 123,132 and 321

Theorem

$$231_n = (n-1)!!\left\lceil \frac{n-3}{2} \right\rceil + (n-2)!!\left\lceil \frac{n-2}{2} \right\rceil,$$

$$\begin{aligned}312_n = {} & (n-1)!!\left(\frac{(-1)^{n-1}+n-3}{4} + \frac{1}{2} \sum_{\substack{k=1 \\ k \neq n \mod 2}}^{n-1} \frac{1}{k} \right) \\ & + (n-2)!!\left(\frac{(-1)^{n}+n-4}{4} + \frac{1}{2} \sum_{\substack{k=1 \\ k = n \mod 2}}^{n-2} \frac{1}{k} \right),\end{aligned}$$

$$213_n = (n-2)((n-1)!! + (n-2)!!) - 231_n - 312_n.$$

Definition

Let π be a permutation. We introduce a standard form for writing π:

1. Each cycle is written with its least element first.
2. The cycles are written in decreasing order of their least element.

The Foata transform $\hat{\pi}$ is the permutation obtained from π by writing it in standard form and by erasing the parentheses separating the cycles.

Example: The involution $\pi = 732458169$ has standard form

$$\pi = (9)(6\ 8)(5)(4)(2\ 3)(1\ 7),$$

so $\hat{\pi} = 968542317$.

The Foata transform

Theorem - *Claesson (2001)*

$\pi \mapsto \hat{\pi}$ induces a bijection between the set of involutions $\mathcal{I}_n$ and $\mathrm{Av}_n(123, 132)$.

It suffices to count patterns in the involutions!

Patterns in the involutions

Pattern in $\mathrm{Av}_n(123, 132)$	Pattern in $\mathcal{I}_n$, with $a < b < c$
321	$(c)(b)(a)$ or $(c)(b)(a\ \star)$ or $(\star\ c)(b)(a)$
231	$(b\ c)(a)$ or $(b\ c)(a\ \star)$
213	$(b)(a\ c)$ or $(\star\ b)(a\ c)$
312	$(c)(a\ b)$ or $(\star\ c)(a\ b)$

Table: The correspondence between patterns in $\mathrm{Av}_n(123, 132)$ and $\mathcal{I}_n$.

There are few fixed points in the involutions

Lemma

Let fp_n be the total number of fixed points in $\mathcal{I}_n$. Then

$$\frac{\mathrm{fp}_n}{|\mathcal{I}_n|} \underset{n \to \infty}{\sim} \sqrt{n}.$$

It suffices to count fixed point-free patterns in the involutions!

Fixed point-free patterns in the involutions

Pattern in $\mathrm{Av}_n(123, 132)$	Fixed point-free pattern in $\mathcal{I}_n$
321	$\emptyset$
231	$(b\ c)(a\ \star)$
213	$(\star\ b)(a\ c)$
312	$(\star\ c)(a\ b)$

Table: The correspondence between fixed point-free patterns in $\mathrm{Av}_n(123, 132)$ and $\mathcal{I}_n$.

Proposition

$\mathrm{pop}_{17}(321) = 0$ and $\mathrm{pop}_{17}(231) = 1/2$.

Popularity of 213

$$213 \longleftrightarrow (\star\ b)(a\ c) = (b\ c)(a\ d) \longleftrightarrow 2314$$

Proposition

Let $G(z) = \sum_{n=4}^{+\infty} \frac{2314_n}{n!} z^n$ be the EGF of $(2314_n)_{n \geq 4}$. Then

$$G(z) = \frac{e^{\frac{(1+z)^2}{2}}}{2} \int_0^z e^{-\frac{(1+t)^2}{2}} dt + \frac{z(z-2)e^{z+\frac{z^2}{2}}}{4}.$$

$$\frac{2314_n}{n!} = [z^n]\frac{e^{\frac{(1+z)^2}{2}}}{2}\int_0^z e^{-\frac{(1+t)^2}{2}}\,dt \quad + [z^n]\frac{z(z-2)e^{z+\frac{z^2}{2}}}{4}.$$

saddle point bound

asymptotic analysis

$$= o\left(\frac{n|\mathcal{I}_n|}{n!}\right) \qquad\qquad \sim \frac{n|\mathcal{I}_n|}{4n!}$$

Corollary

$\text{pop}_{17}(312) = \text{pop}_{17}(213) = 1/4$.

Recap

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Figure: The asymptotic popularity of patterns among 18 avoidance classes.

Open questions

- Can we solve Classes 10, 12, 13, 14 and 15?

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- What happens when we avoid only one consecutive pattern of size 3?
- Can we find a set of patterns, avoiding which we will obtain an irrational asymptotic popularity for some remaining pattern?
- Does the asymptotic popularity always exist? If not, can we characterize patterns for which this limit exists?
- What about the same problem for classical patterns? (*Bóna, Homberger, Janson*)