Enumerating Grid Classes Using MSOL

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Graphs
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Optimization
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Theoretical
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Permutation Patterns 2025 July 8, 2025

Automaton connection

1. Geometric grid classes can be represented[1] by WS1S formulas (Braunfeld 2024).
   $\Rightarrow$ Python script to generate the formula.
2. WS1S formulas can be represented by (deterministic) finite automatons.
   $\Rightarrow$ MONA tool.
3. Obtain the generating function counting strings of length k accepted by a DFA.
   $\Rightarrow$ Solve system of linear equations (we use SageMath).

[1] We need representations to be 1:1 and size preserving.

How does the formula look?

Used version of MSO

- Objects: integers $1, \ldots, n$.
- 1st order variables.
- 2nd order variables: only unary relations (aka sets).
- We have $x \in X$ and $x < y$ but not $x + y$.

We want

Two linear orders: `left_of`(x, y) and `below_of`(x, y).

How does the formula look?

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Two linear orders: `left_of`(x, y) and `below_of`(x, y).

- MSO cannot encode arbitrary linear order directly (in a size preserving way) – there is too many of them.
- We must choose some small subclass of permutations…

1st option: Insertion encoding (simplified)

- We fix k – the number of insertion points.
- Inserting at a point inserts before it.
- We insert values from 1 to n.
- Encoding: partition P_i for $i \in \{1, \ldots, k\}$.
  E.g., for permutation 24513 we get $P_1 = \{2, 4, 5\}$, $P_2 = \{1, 3\}$.
- The linear orders:

$$\texttt{below\_of}(x, y) := x < y$$

$$\texttt{left\_of}(x, y) := \begin{cases} x < y & \text{if } \exists i : x \in P_i \wedge y \in P_i \\ i < j & \text{if } x \in P_i \wedge y \in P_j \text{ and } i \neq j \end{cases}$$

- Size preserving but not 1:1.
- Taking minimal representation helps.

2nd option: Grid encoding

- Fix a matrix $M \in \{-1, 0, 1\}^{r \times c}$:
  - -1: decreasing,
  - 0: empty,
  - 1: increasing.
- We number nonzero entries $1, \ldots, k$.
- Encoding: partition A_i for $i \in \{1, \ldots, k\}$.
- Signs for rows and columns need to be selected so $r_i c_j = M_{i,j}$ for all $M_{i,j} \neq 0$.
- `below_of` and `left_of` similar to `left_of` of insertion encoding but account for signs.
- Again size preserving but not 1:1.
- Idea of the fix is the same but much more complicated.
- Worked out by Braunfeld in 2024.

What about cyclic classes?

- Braunfeld's paper works for geometric grid classes.
- We implement only acyclic grid classes.
- Why?

- Existence of cycles requires extra checks – the formula is bigger.
- Even without them, computation explodes above 5 non-zero entries (partly due to limitations of MONA).
- So it's planned but low priority…

Results

Class	M	Generating function
L_1	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	$-\frac{2x^4 - 6x^3 + 4x^2 - x}{2x^4 - 9x^3 + 12x^2 - 6x + 1}$
L_4	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	$-\frac{x^4 - 4x^3 + 3x^2 - x}{(1-x)^2(1-3x+x^2)}$
T_5	$\begin{pmatrix} -1 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix}$	$\frac{(14x^5 - 56x^4 + 71x^3 - 39x^2 + 10x - 1)x}{(2x^2 - 4x + 1)(x^2 - 3x + 1)(3x - 1)(x - 1)^2}$
J_3	$\begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$-\frac{(3x^4 - 13x^3 + 17x^2 - 7x + 1)x}{(2x^2 - 4x + 1)(x^2 - 3x + 1)(2x - 1)}$
J_{10}	$\begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 0 \end{pmatrix}$	$-\frac{(3x^5 - 4x^4 - 15x^3 + 18x^2 - 7x + 1)x}{(x^2 - 3x + 1)(3x - 1)(2x - 1)(x - 1)}$
W_{10}	$\begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$	$-\frac{(x^5 - 7x^4 + 19x^3 - 18x^2 + 7x - 1)x}{(x^3 - 6x^2 + 5x - 1)(x^2 - 3x + 1)(x - 1)}$

Example

Source code

```
https://github.com/PitelVonSacek/PerMSO
```

Input file (`ex.yaml`)

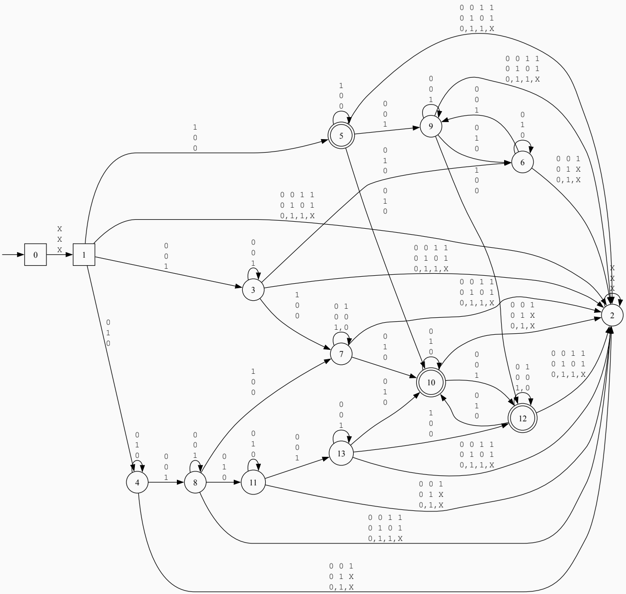
```
class: [[ 1, 1 ],
        [ 0, 1 ]]
```

Shell

```
MONA=ex.mona AUTOMATON=ex.auto EXPAND=30 \
    ./process.sh < ex.yaml
```

- Output generating function and its first 30 values.
- Saves MSOL formula to `ex.mona` and the resulting automaton to `ex.auto`.
- The resulting automaton has $13 + 1$ states.

Example – DFA



X X X
1 0 0
0 0 1
0 1 0
0 0 1 1 0 1 0 1 0,1,1,X
0 0 1 0 1 X 0,1,X
0 1 0
0 0 1
1 0 0
0 1 0
0 1 0 0 1,0
0 1 0 0 1,0
X X X
0
1
2
3
4
5
6
7
8
9
10
11
12
13

Future work

- Possibly better encoding of grid classes.
- MONA improvements:
  - Increase limit on the number of nodes (currently 16M).
  - Experiment with optimizations – notably with the order of variables.
  - Full rewrite easier than hacking MONA code.
  - In progress in very early stage.
- Generic basis calculation (so it supports extra conditions).
- Cyclic geometric grid classes.