The well quasi-order problem for combinatorial structures under the consecutive order

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based on work with Nik Ruškuc

July 9, 2025

Introduction

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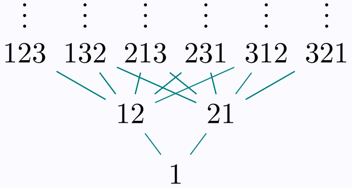

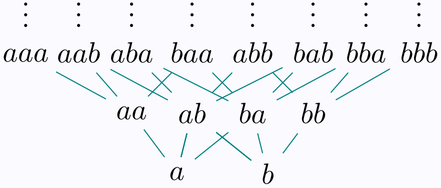

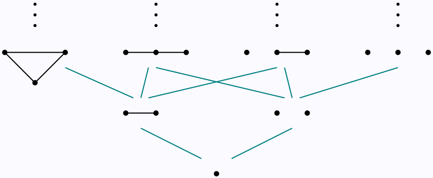

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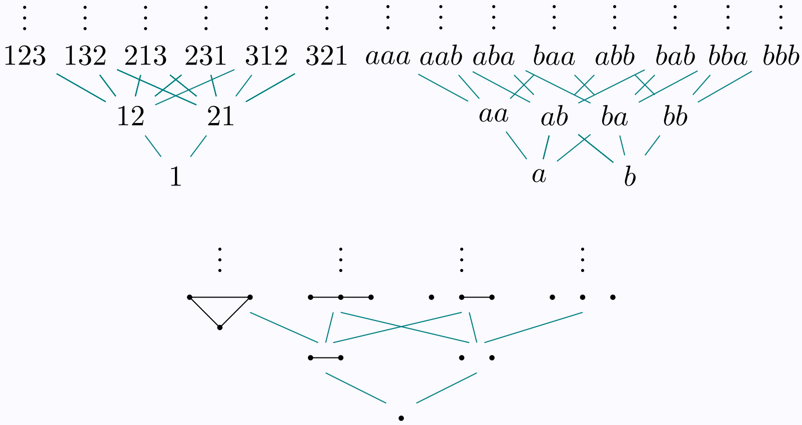

We'll be interested in structural properties of these posets.

Well quasi-order

Definition

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Eg. The set of increasing permutations is wqo as it forms a chain

$$1 \leq 12 \leq 123 \leq \ldots$$

so there are no antichains at all.

Avoidance sets

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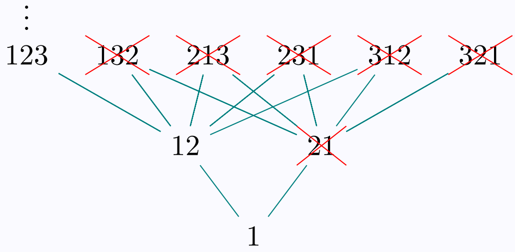
Example

Av(21) is given by:



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Note: if $(C, \leq)$ is wqo, its avoidance sets are also wqo so the wqo problem is trivially decidable.

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For instance, wqo posets are precisely those with countably many downward closed subsets (ideals) [Huczynska & Ruškuc, 2015].

The wqo problem asks not only whether an individual avoidance set is 'tame' or 'wild', but whether there is a clear demarkation between the 'tame' and 'wild' avoidance sets [Cherlin, 2011].

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The wqo problem is decidable for permutations under the consecutive order (McDevitt & Ruškuc, 2021).

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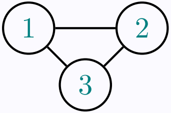

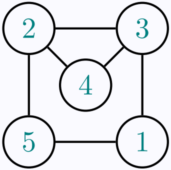

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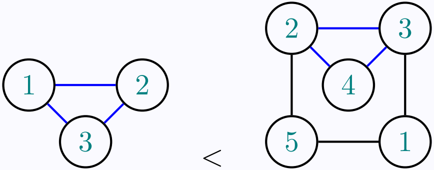

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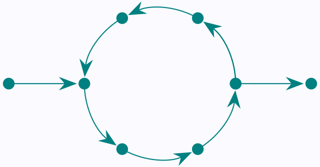

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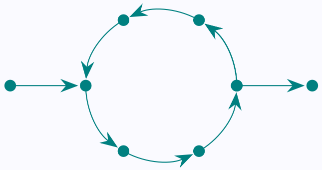

Theorem (McDevitt & Ruškuc, 2021)

The set of paths of a finite digraph G under the subpath order is wqo if and only if G contains no in-out cycles.

Factor graphs

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We will have a running example of permutations.

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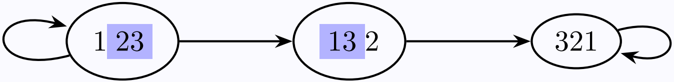

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The path traced by a structure σ will be denoted $\Pi(\sigma)$.

Paths can be traced by more than one structure

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It turns out that the wqo problem is decidable if all or no paths in a factor graph are ambiguous.

Antichains from ambiguous cycles

Lemma

If Γ_C *contains an ambiguous cycle,* C *is not wqo.*

Antichains from ambiguous cycles

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If Γ_C contains an ambiguous cycle, C is not wqo.

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If all paths in Γ_C are ambiguous, C is wqo if and only if Γ_C contains no cycles, i.e. C is finite.

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Theorem

The wqo problem is decidable for the following structures under consecutive orders:

1. *Graphs;*
2. *Digraphs;*
3. *Tournaments;*
4. *n-ary relations;*
5. *Collections of n binary relations.*

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Lemma

If Γ_C contains no ambiguous paths, C is wqo if and only if Γ_C contains no in-out cycles, so the wqo problem is decidable.

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We have already seen that in-out cycles and ambiguous cycles yield infinite antichains. Can infinite antichains arise in other ways? Yes!

Permutations - splittable pairs

Consider the paths that start at 132, go to 321, and then wind round the loop $n \geq 1$ times.

Permutations - splittable pairs

$$\begin{tikzcd} 123 \arrow[loop left] \arrow[r] & 132 \arrow[r] & 321 \arrow[loop right] \end{tikzcd}$$

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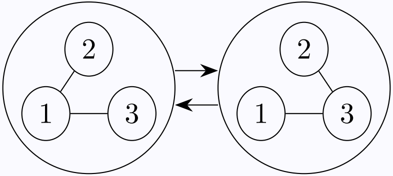
Theorem: An avoidance set C of permutations is wqo if and only if Γ_C contains no in-out cycles, ambiguous cycles or splittable pairs. (McDevitt & Ruškuc, 2021)

WQO for equivalence relations

Theorem: An avoidance set C of equivalence relations is wqo if and only if Γ_C contains no in-out cycles or ambiguous cycles.
(VI & N. Ruškuc, 2024)

Limitations of factor graphs

Consider the following factor graph for forests under the consecutive order:



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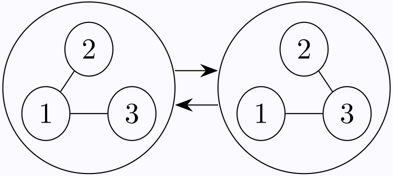

The path from the left vertex to the right vertex has associated structure:

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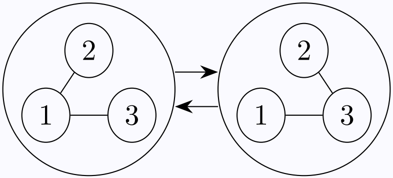
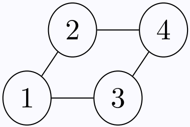
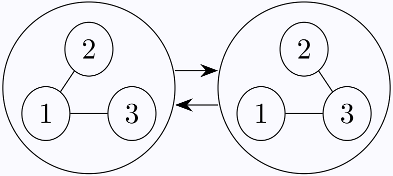


The path from the left vertex to the right vertex has associated structure:



Limitations of factor graphs

Consider the following factor graph for forests under the consecutive order:



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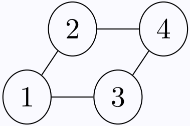

This is not a forest! So not every path corresponds to a structure in this avoidance set.

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The wqo problem is decidable for forests under the consecutive order (VI & N. Ruškuc).

Comparisons

Comparisons

Structure	Conditions on Γ_C for wqo
Words (McDevitt & Ruškuc, 2021)	no in-out cycles
Equivalence relations	no in-out cycles no ambiguous cycles
Permutations (McDevitt & Ruškuc, 2021) Permutations with equivalence relations (ongoing)	no in-out cycles no ambiguous cycles no splittable pairs
Graphs Digraphs Tournaments n-ary relations n binary relations	no (ambiguous) cycles

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- Is there an overarching picture for consecutive orders behind these results?
- Moving away from consecutive orders, can we answer the wqo problem for orders ‘between’ consecutive and non-consecutive orders?
- The wqo problem for permutations under the non-consecutive order remains open.

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Questions

Thank you for listening!