$\mathcal{V}$-antichains and The Realisability Problem

Ben Jarvis

Based on joint work with Robert Brignall

7th July 2025

Outline of talk

- At PP 2024 I used $\mathcal{V}$-antichains to conjecture a classification of 'small' antichains (those with growth rate smaller than $\nu_{\mathcal{L}} \approx 3.28277$.)

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- Finally, I will introduce a connection with an open problem regarding binary factorial languages, which I dub The Realisability Problem.

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- Despite this, relatively few (genuinely distinct) infinite antichains of permutations are known, especially at small growth rates.

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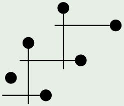
Chain of Oscillations



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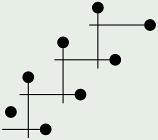

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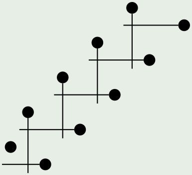

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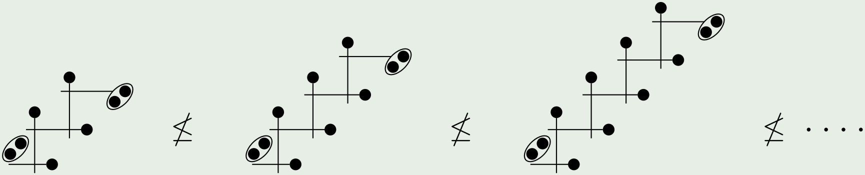

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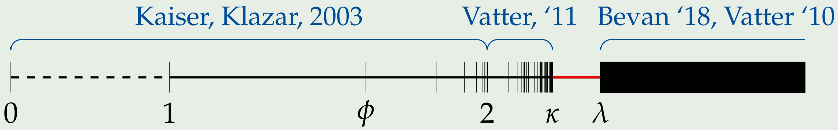
Explains phase-transitions at $\kappa \approx 2.206$ and $\lambda \approx 2.357$ in growth-rate diagram:



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- This suggests that the key to understanding the deeper structure of the set of growth rates of permutation classes may be to understand where infinite antichains appear.
- Infinite antichains are very difficult to find, with relatively few being known. They seem especially rare for small growth rates (smaller than about 3.5). Can we classify 'small' antichains?

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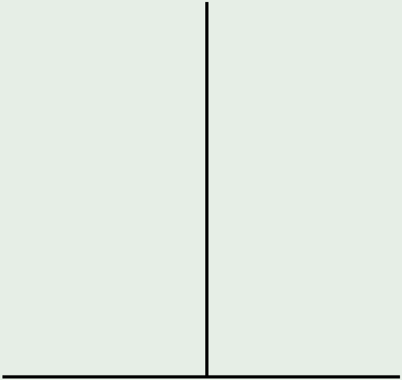
- Thus we will end up with a very large example class of antichains, whose growth rates we can find with relative ease.
- In particular, using binary sequences with low complexity will allow us to construct infinite antichains with very low growth rates. We shall use these to conjecture a complete classification of 'small' infinite antichains.

The *second*-smallest infinite antichain?

We construct a permutation class $\mathcal{V}$ and show that an infinite antichain appears at the same growth rate:

Constructing the permutation class $\mathcal{V}$



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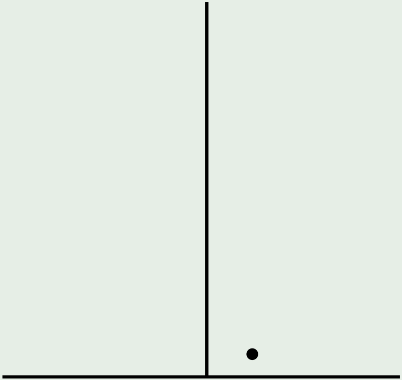

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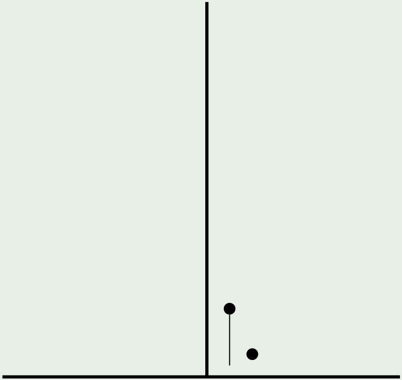

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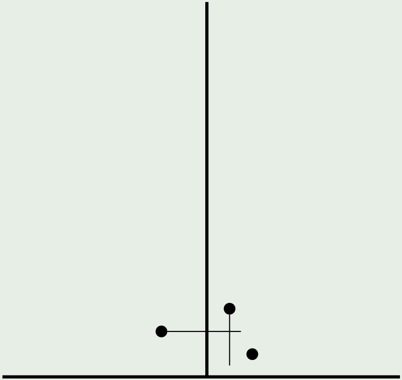

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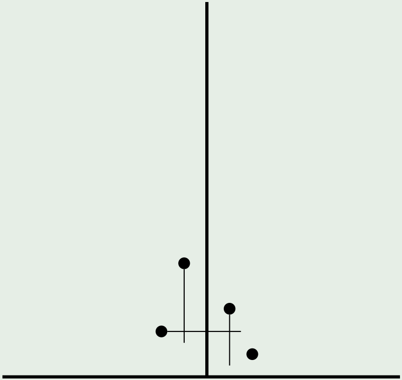

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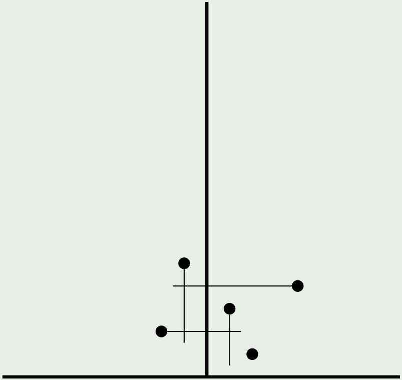

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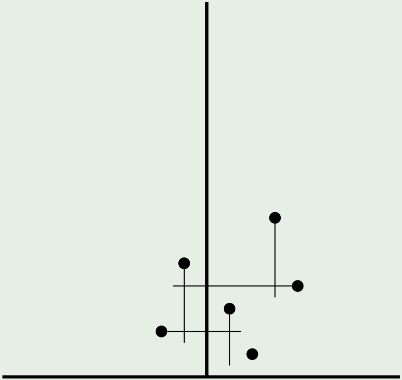

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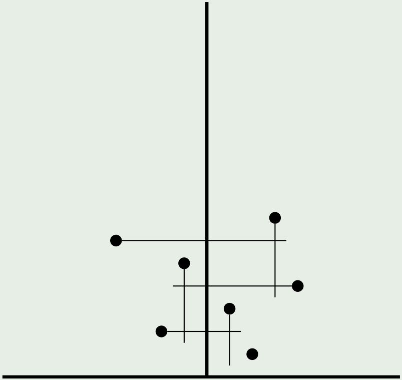

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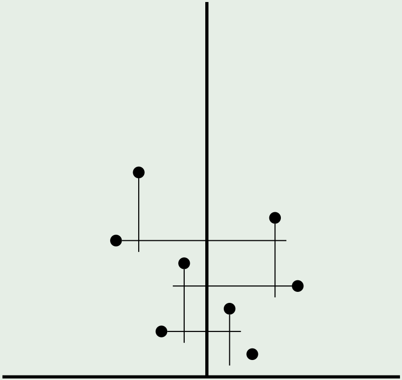

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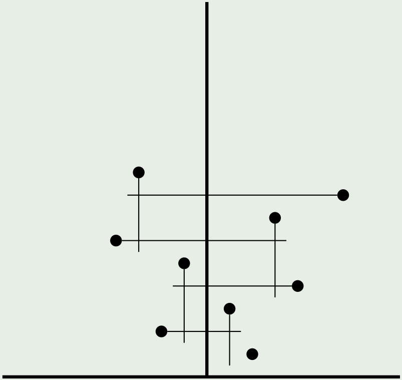

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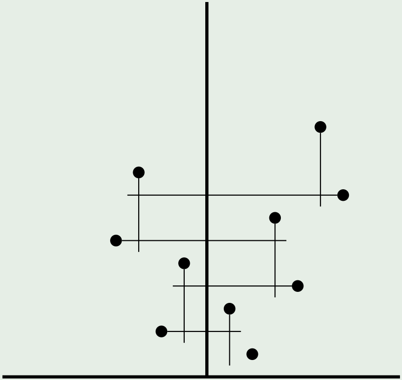

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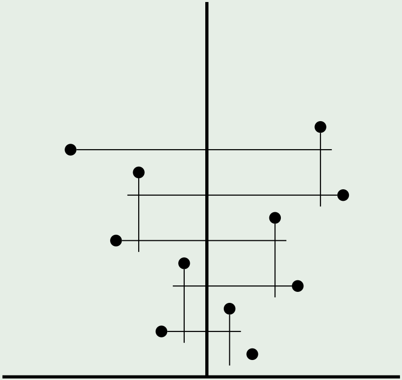

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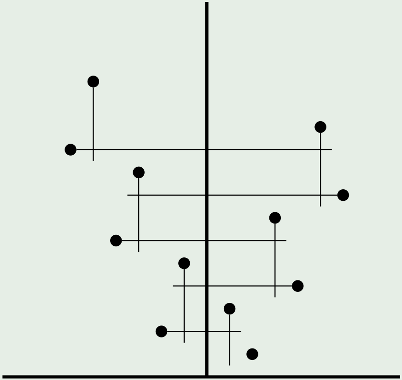

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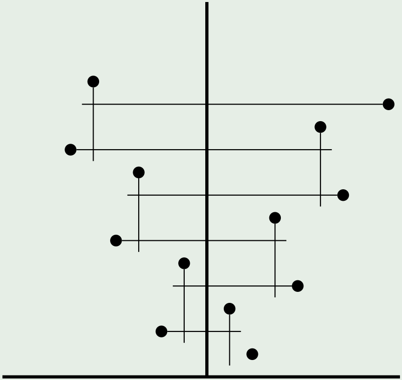

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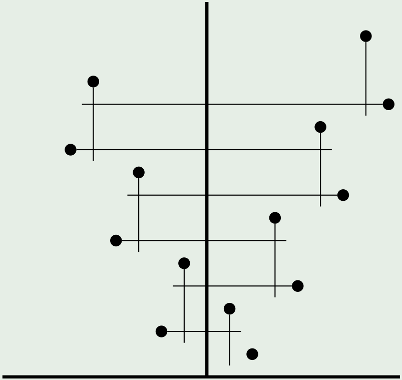

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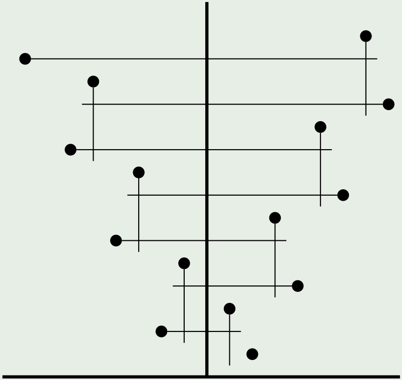

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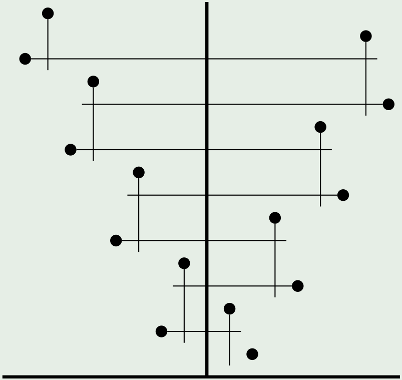

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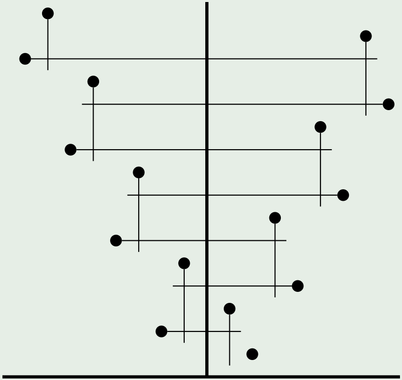

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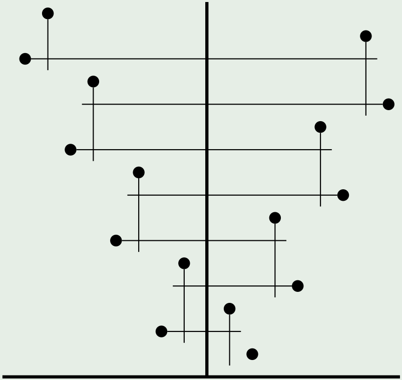

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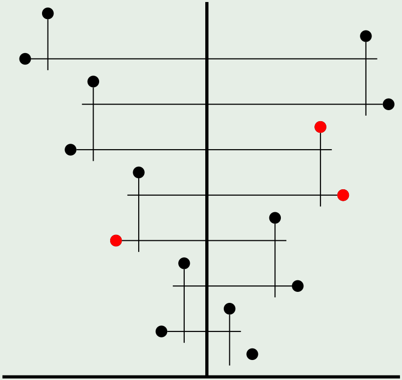

Examples:

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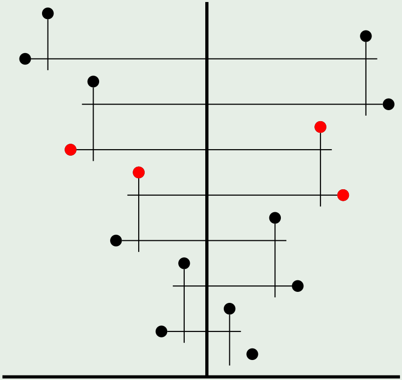

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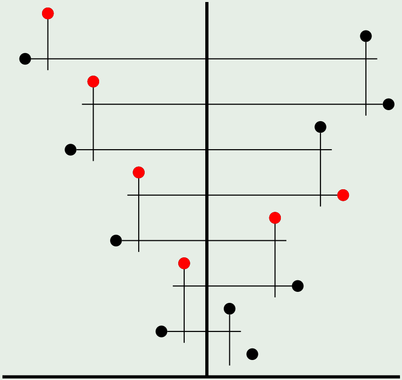

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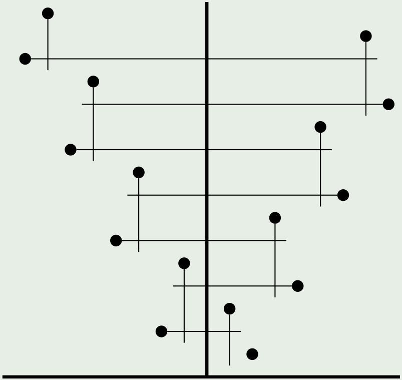

Examples:

- $132 \in \mathcal{V}$
- $3241 \in \mathcal{V}$
- $654123 \in \mathcal{V}$
- $12543 \notin \mathcal{V}$

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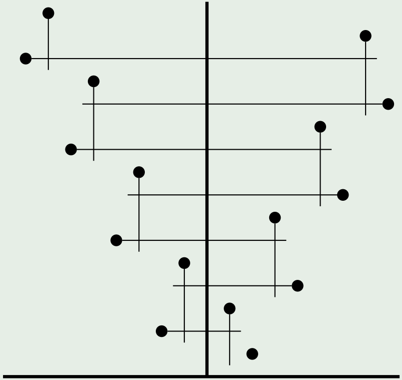
Counting the class $\mathcal{V}$

Enumerating $\mathcal{V}$



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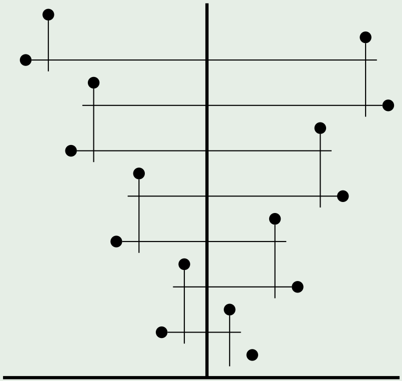

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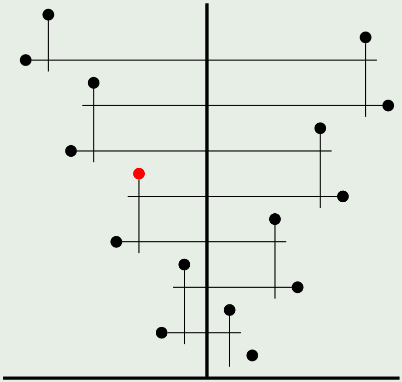

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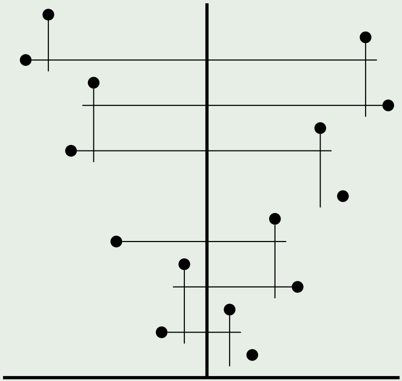

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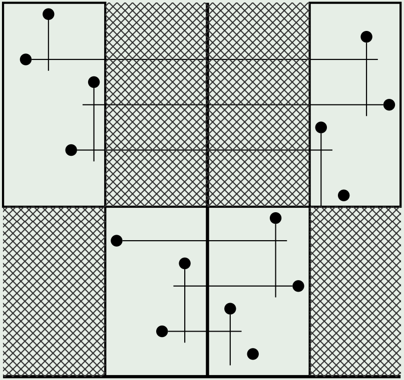

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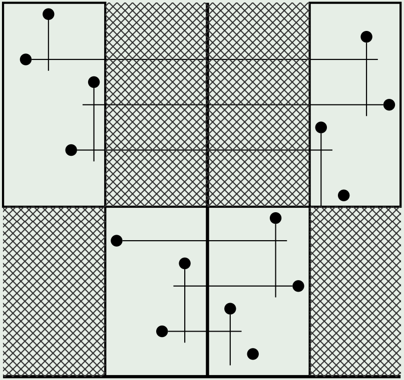

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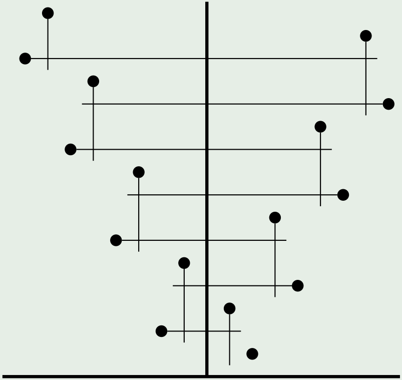
- As soon as we remove an interior point of a $\mathcal{V}$-permutation it decomposes into the 'sum' of two contiguous chunks ('$\boxplus$-indecomposables')
- **Idea for enumerating $\mathcal{V}^\circ$:** find g.f. $g(z)$ of contiguous chunks first (easy), then 'glue' these together with the sequent operator: $f(z) = \frac{1}{1-g(z)}$

Enumerating $\mathcal{V}$

The Class $\mathcal{V}$



$\mathcal{V}^\circ$ has generating function

$$f(z) = \frac{1-z}{1-3z-2z^4}$$

Hence $\mathcal{V}$ has growth rate $\nu \approx 3.069$

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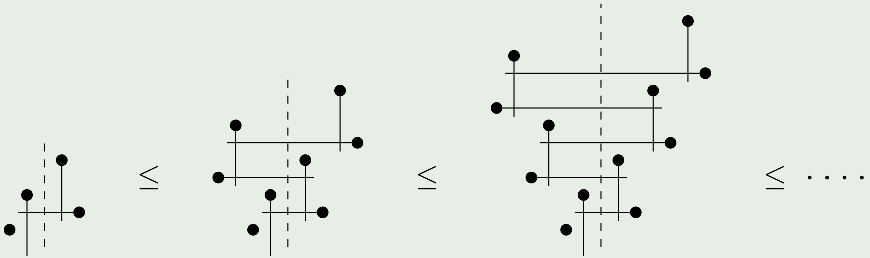

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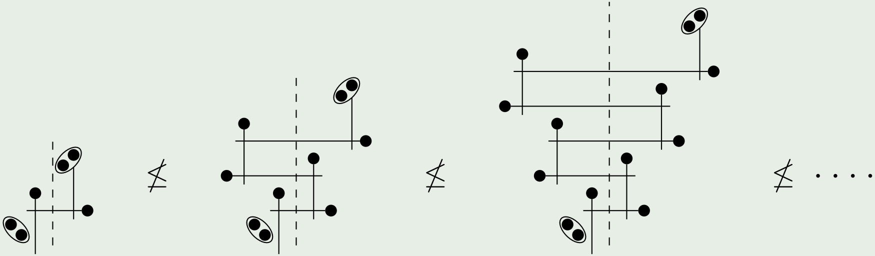

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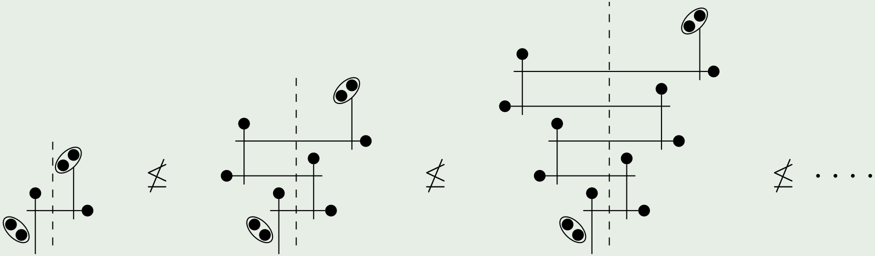

The downward closure of this antichain is sandwiched between the classes $\mathcal{V}$ and $\mathcal{V}^{+2}$, both of which have growth rate $\nu \approx 3.069$.

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- This is *probably* true, though proof is still ongoing
- Most importantly for our purposes, we can generalise the construction of $\mathcal{V}$ to generate an antichain for any given binary sequence...

Turning a binary sequence into a $\mathcal{V}$-class

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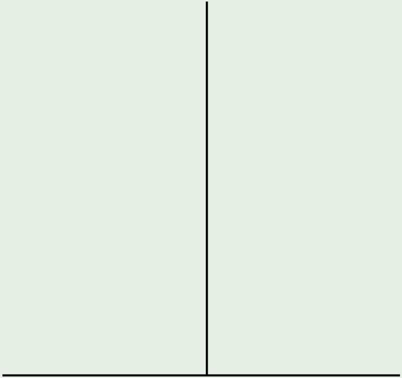

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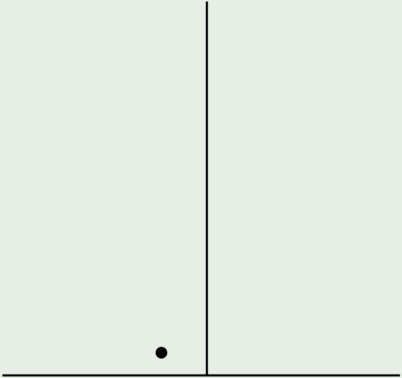

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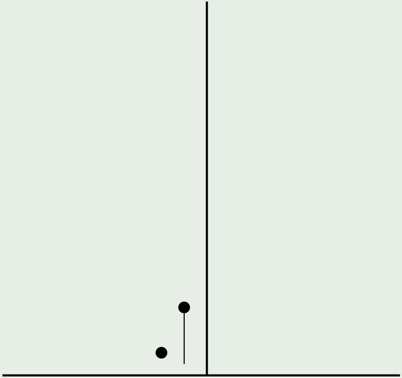

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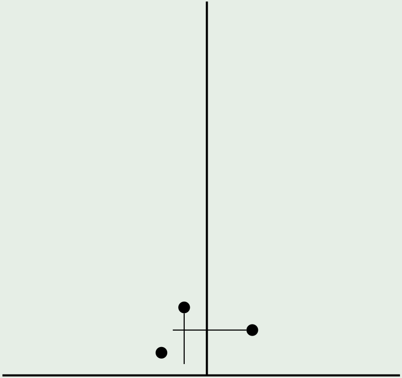

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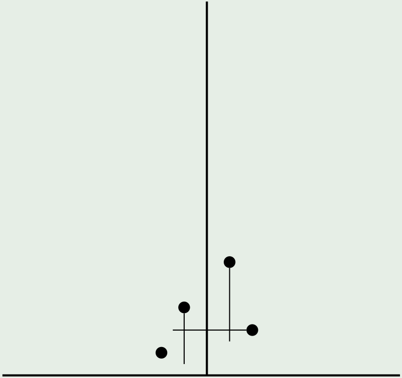

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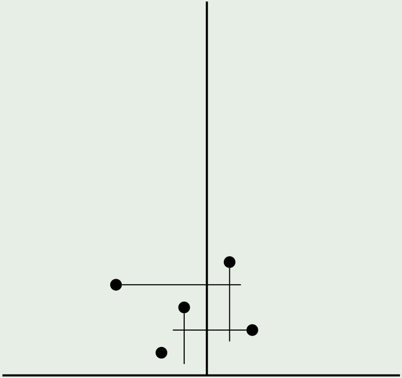

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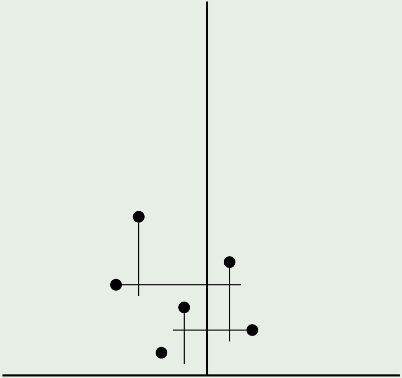

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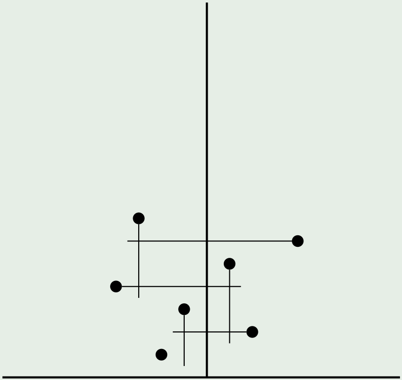

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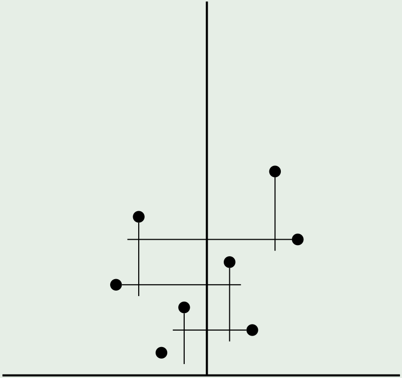

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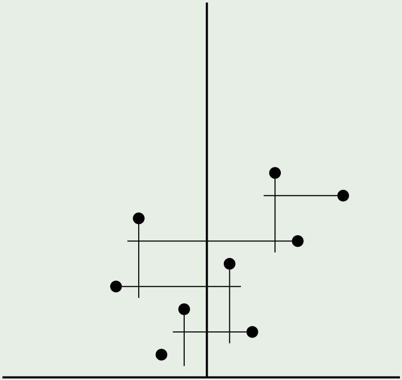

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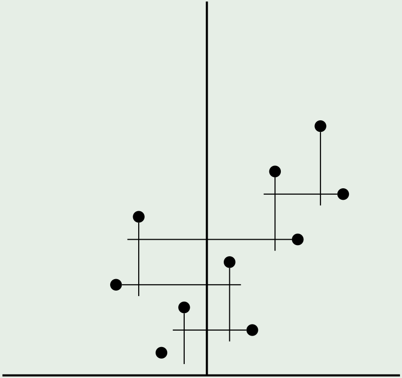

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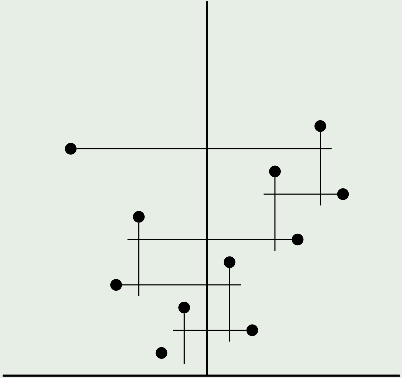

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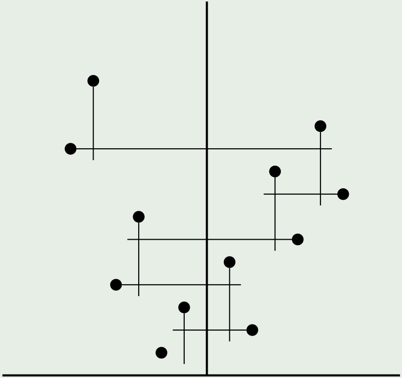

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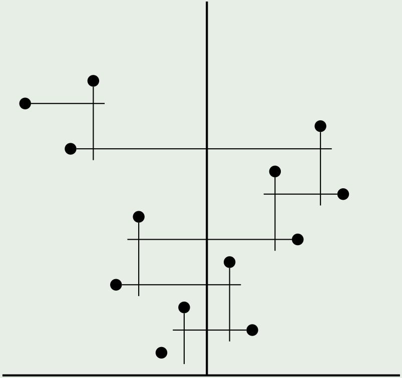

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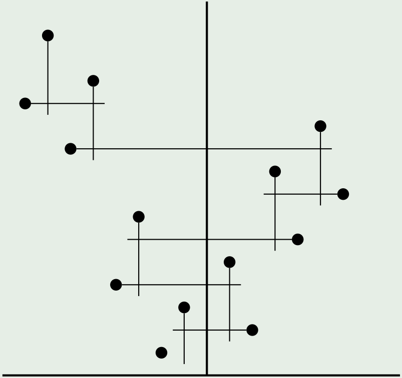

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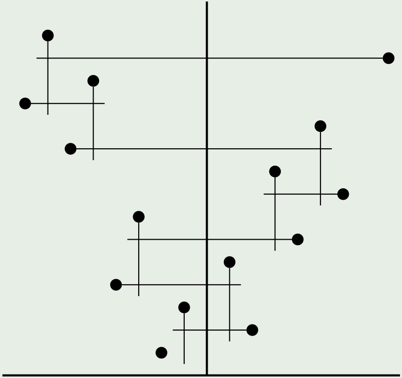

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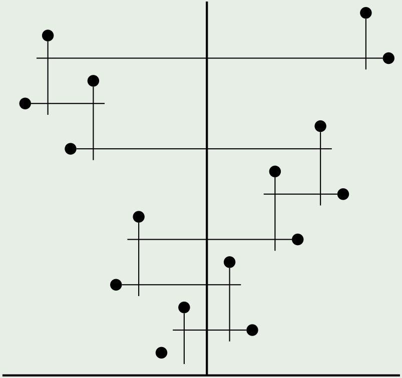

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Enumerating $\mathcal{V}$-classes

- Same idea as for $\mathcal{V}$: count contiguous chunks ('$\boxplus$-indecomposables') and 'glue together' with sequent operator.
- A **recurrent factor** of $\underline{b}$ is a finite binary word which appears as a substring in $\underline{b}$ **infinitely-often**.
- For a given binary sequence $\underline{b}$, let $(q(n))$ denote the **recurrent complexity sequence**: $q(n)$ is the number of recurrent factors of $\underline{b}$ of length n.
- The recurrent complexity sequence $(q(n))$ of $\underline{b}$ allows us to find the growth rate of the corresponding $\mathcal{V}$-class.

Precisely: if $t(z) = \sum_{n=1}^{\infty} q(n) z^n$ and $q(1) = 2$ then the growth rate of $\mathcal{V}_{\underline{b}}$ is the reciprocal of the smallest positive real solution of the equation

$$\frac{(1+z)^2 t(z^2)}{z^2} - 2 - 2z - q(2)z^2 - 2z^3 = 1.$$

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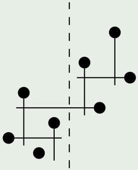

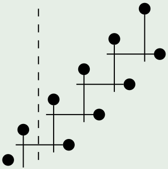

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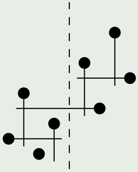

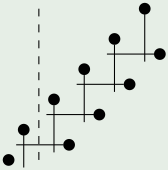

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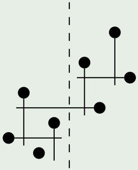

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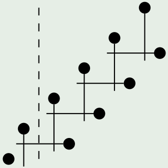

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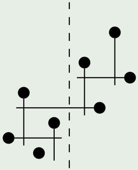

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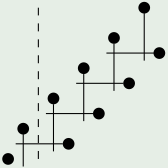

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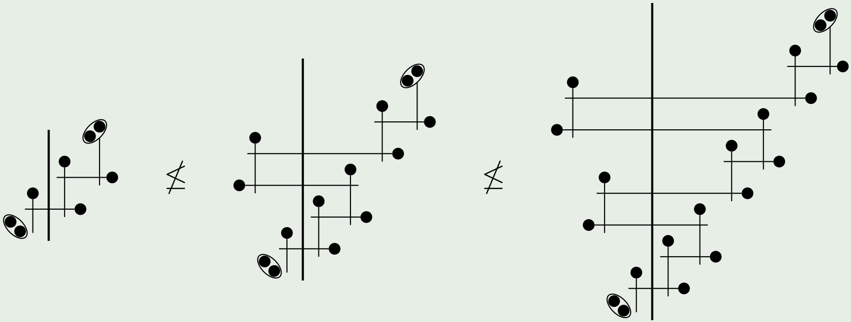

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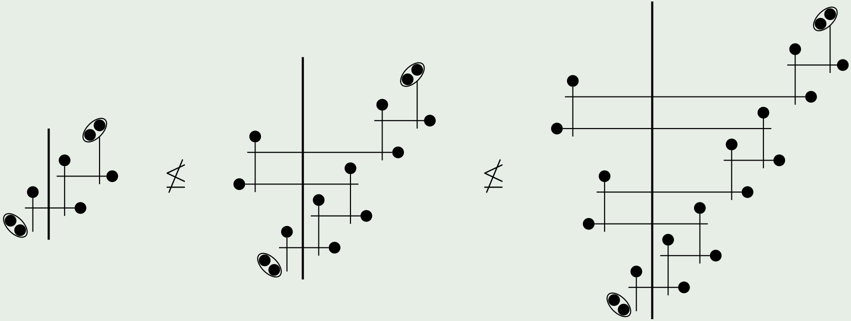

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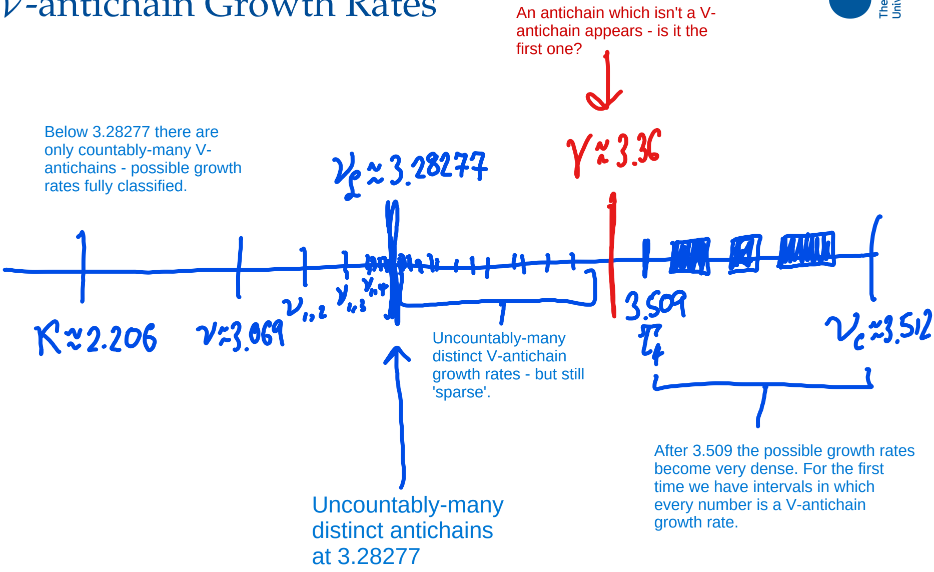
In particular, choosing defining binary sequences with low complexity will generate antichains with low growth rates...

$\mathcal{V}$-antichain Growth Rates



Uncountably-many antichain g.r.s in $(\nu_{\mathcal{L}}, \nu_{\mathcal{L}} + \varepsilon)$

Recall that $\nu_{\mathcal{L}} \approx 3.28277$. This is the first growth rate at which there are uncountably many genuinely distinct $\mathcal{V}$-antichains. Let $\varepsilon > 0$:

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- Enumerate the language $Av(B)$ and substitute generating function into growth rate operator.

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- Let $\{0,1\}^*$ denote the set of all finite binary words and let B be a subset of $\{0,1\}^*$, none of whose elements is a factor (ie., contiguous subsequence) of any other.
- We take B to be the antidictionary of a factorial language: $Av(B)$ is the set of binary words which do not contain any element of B as a factor.
- **Is there an infinite binary sequence $\underline{b}$ whose set of recurrent factors is precisely $Av(B)$?**

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- $Av(11)$ is realisable: Write $Av(11) = \{b_1, b_2, b_3, \dots\}$. Then

$$\underline{b} = b_1 0 b_1 0 b_2 0 b_1 0 b_2 0 b_3 0 b_1 \dots \quad \text{works.}$$

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- $Av(0101011, 00)$ is not realisable: consider right-extensions of 010101...

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A complete solution to this problem would **likely** enable us to construct longer intervals of growth rates of (trim) antichains.

A Mysterious Fact:

It is certainly not the case that every avoidance set $Av(B)$ of binary words is realisable. But every **unbounded** avoidance set $Av(B)$ of which I am aware has **precisely the same enumeration sequence as a realisable language.** Is this a coincidence or a general result? If true it would allow the construction of wider intervals of trim antichains.

Examples:

$$Av(01) \cong Av(11, 101, 1001, 10001, \ldots)$$

$$Av(001) \cong Av(101, 10001, 1000001, 100000001, \ldots)$$

$$Av(010, 11) \cong Av(11, 101, 1001, 0000)$$

non-realisable | Realisable

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