Topology of permutation patterns

Krishna Menon
KTH Royal Institute of Technology, Sweden

Permutation Patterns 2025

Ongoing joint work with Priyavrat Deshpande and Anurag Singh

Permutation $\pi \quad \rightarrow \quad$ Simplicial complex X_π

Structural (pattern) properties of π

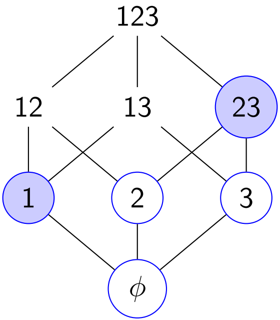
$$\updownarrow$$

Topological properties of X_π

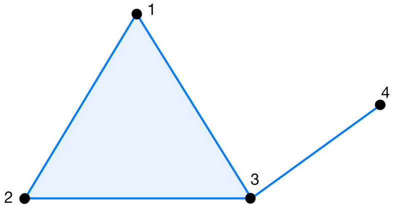
Definition

A simplicial complex on the vertex set $[n]$ is a collection X of its subsets which is closed under inclusion and contains all singletons.

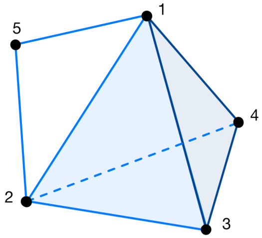
- The sets in X are called faces.
- A maximal face in X is called a facet.



Geometric view



Facets $= 123, 34$



Facets $= 1234, 15, 25$

Definition

For a permutation $\pi = \pi_1\pi_2\cdots\pi_n$, the simplicial complex X_π has faces

$$\{i_0, i_1, \ldots, i_k\}_< \quad \text{such that} \quad \pi_{i_0} < \pi_{i_1} < \cdots < \pi_{i_k}.$$

$$\pi = 3\ 2\ 5\ 4\ 1\ 7\ 6$$

$$137, 246, 56, \ldots \in X_\pi$$

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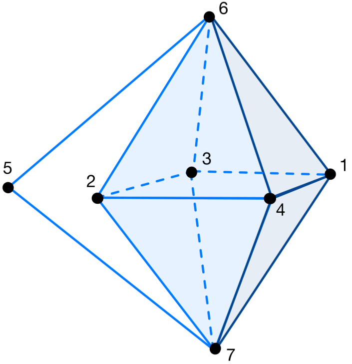
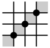
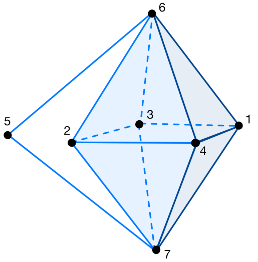
X_π for $\pi = 3254176$

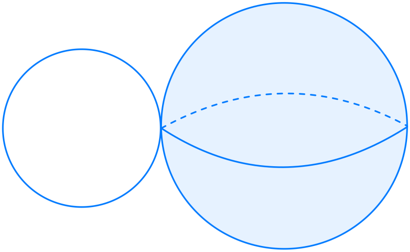


Facets correspond to , , , . . .



X_π

$\simeq$



$S^1 \vee S^2$

Theorem (Przytycki and Silvero, 2018)

For any permutation π, X_π is homotopic to a wedge of spheres, i.e.,*

$$X_\pi \simeq S^{k_1} \vee S^{k_2} \vee \cdots \vee S^{k_m}$$

*Actually disjoint union of wedges of spheres. $X_{\sigma \ominus \tau} = X_\sigma \sqcup X_\tau$.

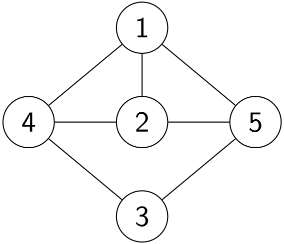
History

- Independence complexes of circle graphs.
  Przytycki and Silvero, 2018
- Complexes of injective words.
  Chacholski, Levi, and Meshulam, 2020
- Topological connectivity of random permutation complexes.
  Meshulam and Moyal, 2024

Graph complexes

Popular topic: Graph $G \to$ Simplicial complex X_G

$4\ 3\ 5\ 1\ 2 \quad \rightsquigarrow$



X_π is the <u>independence complex</u> of the inversion graph of π.

Disclaimer

Priyavrat

$\longleftarrow$ Topological combinatorists $\longrightarrow$

Anurag

Disclaimer

Priyavrat

Topological

combinatorists

Anurag

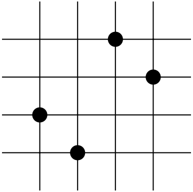
Humble enumerative

combinatorist

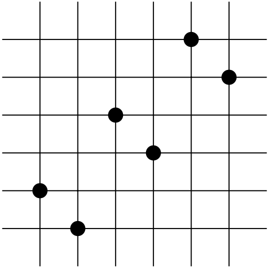
Special patterns

Definition

The cross-pattern of dimension k is $\mathrm{cp}_k := \bigoplus^{k+1} 21$.

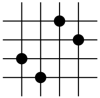


$\mathrm{cp}_1 = 2143$

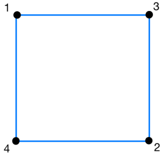


$\mathrm{cp}_2 = 214365$

$X_{\mathrm{cp}_k} \simeq S^k$

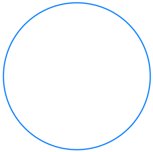


cp_1

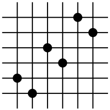


X_{cp_1}

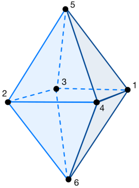
$\simeq$



S^1

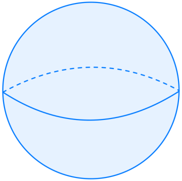


cp_2



X_{cp_2}

$\simeq$



S^2

If $X_\pi \simeq S^{k_1} \vee \cdots \vee S^{k_m}$, then

π contains the patterns $\mathrm{cp}_{k_1}, \ldots, \mathrm{cp}_{k_m}$.

Theorem

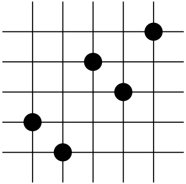
For any $k \geq 1$, if $\pi \in \mathrm{Av}(\mathrm{cp}_k)$, then $\mathrm{hdim}(X_\pi) < k$.

$$\mathrm{hdim}(S^{k_1} \vee \cdots \vee S^{k_m}) = \max\{k_1, \ldots, k_m\}$$

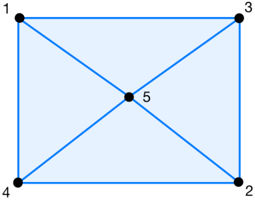

So we have good news,
and we have bad news.
My nana always said, "Bad news
first, because the good news
is probably a lie."

Bad news

The converse is not true.

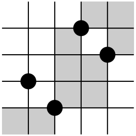


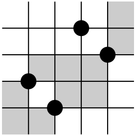
$\pi = 21435$



$X_\pi \simeq \bullet$

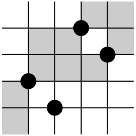
Good news

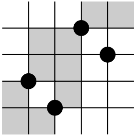




Theorem

If $X_\pi \simeq S^{k_1} \vee \cdots \vee S^{k_m}$ and π has a 'certain type' of occurrence of cp_k, then $k \in \{k_1, \ldots, k_m\}$.





Highest hdim

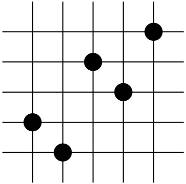
Corollary

For any $\pi \in \mathfrak{S}_n$, $\operatorname{hdim}(X_\pi) \leq \lfloor \frac{n}{2} \rfloor - 1$. *This upper bound is achieved*

- *only by* $\operatorname{cp}_{\frac{n}{2}-1}$, *if n is even and*
- *by precisely* $n^2 - 5\left(\frac{n-1}{2}\right) - 1$ *permutations, if n is odd.*

Questions and a generalization

Question 1



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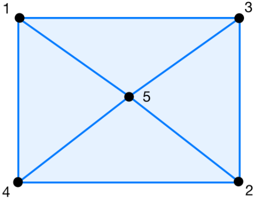
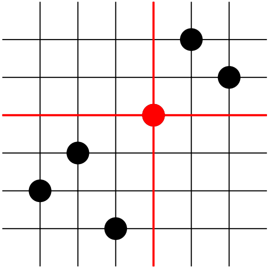


X_π

$\simeq$ $\bullet$

For which permutations π is X_π contractible?

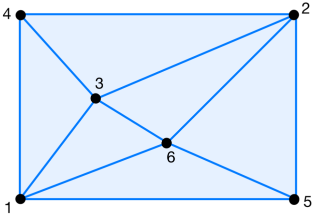
- If π has a strong fixed point, then $X_\pi \simeq \bullet$.



- If $\pi \in \mathrm{Av}(\mathrm{cp}_1)$ and X_π is connected, then $X_\pi \simeq \bullet$.

$\pi = 214635$

Contains cp_1
No strong fixed points



X_π

$\simeq$ $\bullet$

Characterize and count the permutations π for which $X_\pi \simeq \bullet$.

Question 2

Longest increasing subsequences of π

$$\updownarrow$$

Facets of largest cardinality in X_π.

The two extremes:

- Unique facet with largest cardinality.
- All facets have same cardinality.

Question 2

Longest increasing subsequences of π

$$\updownarrow$$

Facets of largest cardinality in X_π.

The two extremes:

- Unique facet with largest cardinality. ULIS problem
- All facets have same cardinality.

Definition

A simplicial complex is pure if all facets have the same cardinality.

X_π is pure if *any* increasing subsequence can be extended to an LIS.

Characterize and count the π for which X_π is pure.

- Questions answered for some classes of permutations.

  Permutations avoiding a size 3 pattern, Grassmannian permutations, Involutions in Av(3412).

- Can ask several other questions based on topology of X_π.

  Number of S^k in homotopy type of X_π, $\mathrm{hdim}(X_\pi)$, when is X_π shellable, vertex-decomposable, etc.

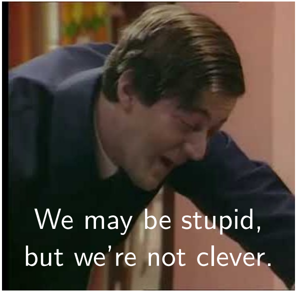
Generalization

Given a pattern σ and permutation π,

$$X_\pi^\sigma := \{\mathtt{I} \mid \pi_{\mathtt{I}} \text{ avoids } \sigma\}.$$

- X_π corresponds to $\sigma = 21$.
- For decreasing patterns δ, we have that X_π^δ is a wedge of spheres.

Thank you!



Also someone who couldn't
think of an end-of-talk joke

References

- W. Chachólski, R. Levi and R. Meshulam, *On the topology of complexes of injective words*, J. Appl. Comput. Topol. (2020).
- R. Meshulam, O. Moyal, *Topological connectivity of random permutation complexes*, arXiv:2406.19022 (2024).
- J. H. Przytycki and M. Silvero, *Homotopy type of circle graph complexes motivated by extreme Khovanov homology*, J. Algebraic Combin. (2018).