Higher dimensional floorplans and Baxter d-permutations

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Permutation patterns, July 9 2025

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Joint work : N. Bonichon and A. Tanasa

LABRI, Univ. Bordeaux

Mosaic floorplans and Baxter permutations

What is a mosaic floorplan ?

Definition

A mosaic floorplan is a partition of a rectangle into interior-disjoint rectangles (blocks) such that :

- No points belong to the boundary of four rectangles (tatami rule).

Structurally equivalent floorplans

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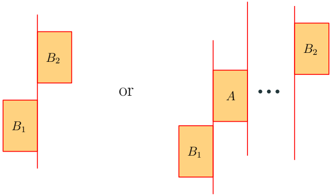
For two blocks B_1 and B_2, B_1 is to the left of B_2 (denoted $B_1 \overset{x}{\twoheadleftarrow} B_2$) if :

- There is a vertical segment containing the left side of B_2 and the right side of B_1
- There is a chain of blocks $b_1 \dots b_r$ between B_1 and B_2 such that this is the case. (transitive closure of the relation above)



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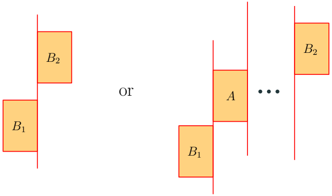
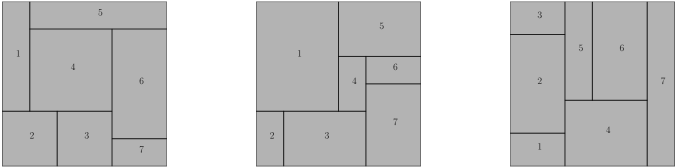


We define similarly, B_1 is above B_2 (denoted $B_1 \overset{y}{\twoheadleftarrow} B_2$).

These relations are called direction relations

Structurally equivalent floorplans

Two floorplans are structurally equivalent if their relations $\overset{x}{\twoheadleftarrow}$ and $\overset{y}{\twoheadleftarrow}$ are the same for all blocks (up to relabeling).



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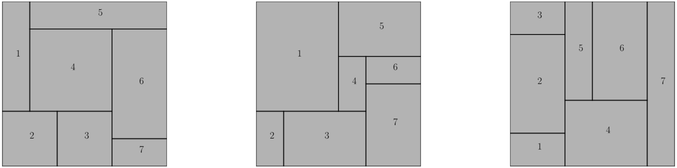
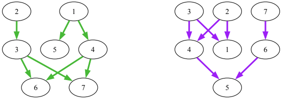


First two floorplans are equivalent.



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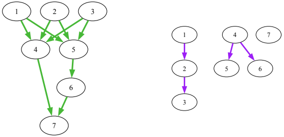
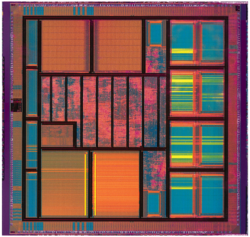
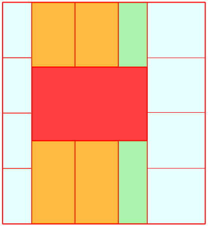


Last floorplan

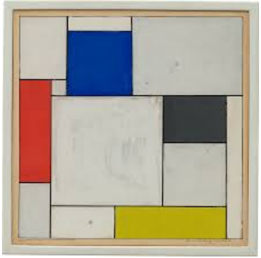
Applications of floorplans

Studied as model for designs of integrated circuits (VLSI).





Also appear in visual art



Composition décentralisée

van Doesburg (1924)

Combinatorics of floorplans

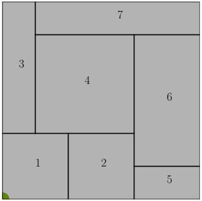
(Weak) mosaic floorplans belongs to the Baxter combinatorial family.

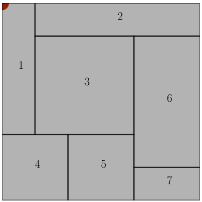
Counted by the Baxter numbers (OEIS A001181)

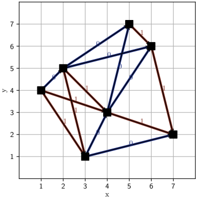
$$|B_n| = \sum_{k=1}^{n} \frac{\binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}}{\binom{n+1}{1}\binom{n+1}{2}} .$$

They are in bijection with many other structures :

Pattern avoiding permutations (Ackerman, Barequet, Pinter 2006)







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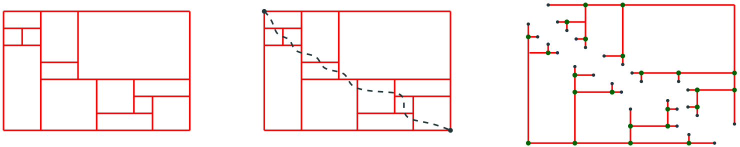
They are in bijection with many other structures :

Twin binary trees (Young, Chu, Shen 2002)



Baxter permutations

Definition

A vincular pattern $\pi|_X$ is a pattern π with k points, together with a list of adjacency conditions X (in $\{1, \ldots, k-1\}$)

An occurrence of π is also an occurrence of $\pi|_X$ if for any k in X, the k^{th} and the $(k+1)^{th}$ points of the occurrence are adjacent w.r.t the x axis.

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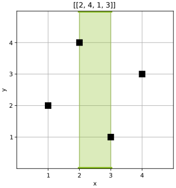
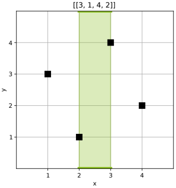
Baxter permutations are the permutations avoiding $2413|_2$ and $3142|_2$.





Baxter permutations

Baxter permutations can also be defined as well-sliced permutations.

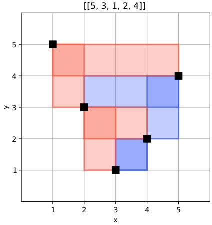
Definition

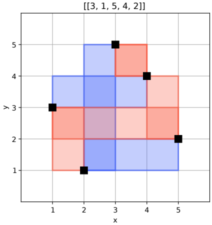
Let p_i and p_j be two adjacent points in a permutation π. The slice of p_i and p_j is defined as the rectangle with p_i and p_j as corners. A slice is positive if it corresponds to an ascent and negative for a descent.

A permutation is well sliced if :

- each vertical slice intersects exactly one horizontal slice (except slices of size 1).
- positive/negative slices intersect only positive/negatives slices.

✓



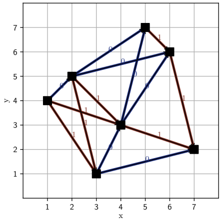


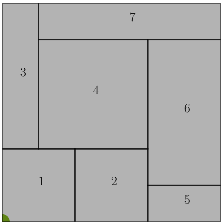
X

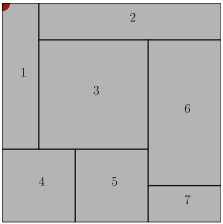
A quick summary of the bijection

Bijection found in Ackerman, Barequet, Pinter 2006

Baxter permutations	mosaic floorplans
points	blocks
relative position of the points	direction relation between the blocks
x-coordinate of the points	peeling order w.r.t bottom left corner
y-coordinate of the points	peeling order w.r.t top left corner



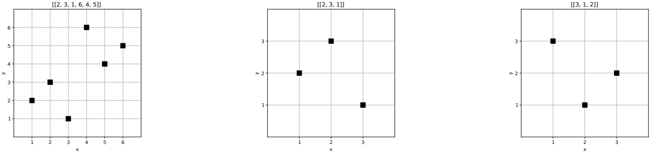




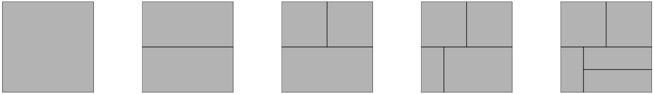
A quick summary of the bijection

This bijection also specializes to a bijection between :

- Separable permutations (2413 and 3142 avoiding permutations)

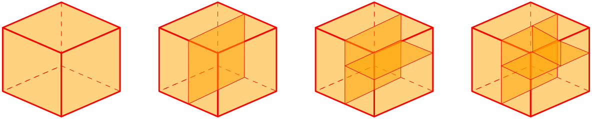


- Slicing floorplans ( and  avoiding mosaic floorplans)



Separable d-permutations and guillotine partitions

In Asinowski & Mansour 2010, a bijection between separable d-permutations and 2^{d-1} dimensional guillotine partitions (generalization of slicing floorplans) was proven.



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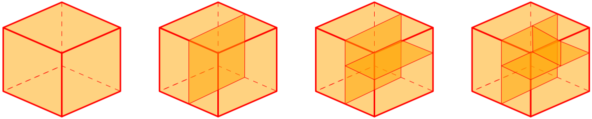
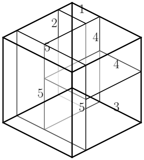
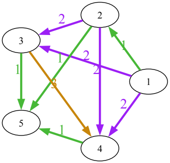
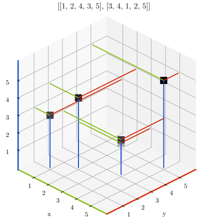


In this work : Generalization of mosaic floorplans and of the bijection with Baxter permutations to higher dimensions.







Higher dimensional permutations

d-permutations : definition

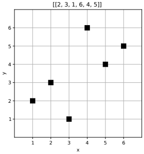
A $d-$permutation is a tuple of $d-1$ permutations, $\Pi = (\pi_1, \ldots, \pi_{d-1})$.
(S_n^{d-1} set of d-permutations with n points).

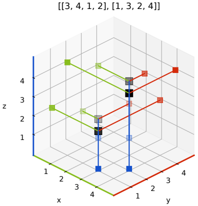
Can be written in matrix form.

Example for a 3-permutation :

$$\Pi = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ \pi_1^1 & \pi_1^2 & \pi_1^3 & \ldots & \pi_1^n \\ \pi_2^1 & \pi_2^2 & \pi_2^3 & \ldots & \pi_2^n \end{pmatrix}$$

Can also be represented by a point set on a d-dimensional grid.





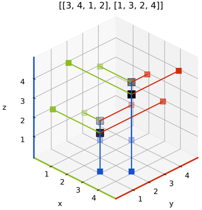
Pattern avoiding d-permutations

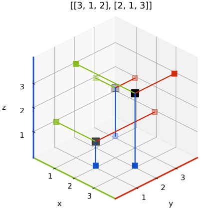
Definition

Let $\sigma \in S_n^{d-1}$ and $\pi \in S_k^{d-1}$ with $k \leq n$.

σ contains the pattern π if there exists a subset of points of σ that is equal (once standardized) to π. A permutation that does not contain a pattern avoids it.

ex : $(3412, 1324)$ contains the pattern $(312, 213)$





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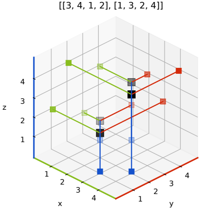
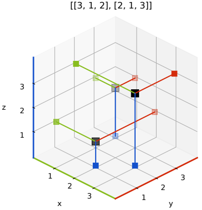
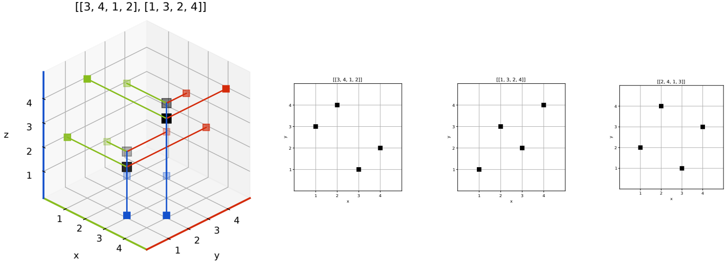




$Sym(\pi)$ = all patterns obtained by applying symmetries of the $d-$grid on π. One can also consider lower dimensional patterns ($\pi \in S_k^{d'-1}$ with $d' < d$).

Definition

Let $\mathbf{i} = i_1 \dots i_{d'}$ be a sequence of indices in $\{0, \dots, d\}$, let also $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_{d-1})$ be a d-permutation of size n. The *projection* on $\mathbf{i}$ of $\boldsymbol{\sigma}$ is the d'−permutation given by $\text{proj}_{\mathbf{i}}(\boldsymbol{\sigma}) = (\sigma_{i_2}\sigma_{i_1}^{-1}, \dots, \sigma_{i_{d'}}\sigma_{i_1}^{-1})$. A projection is *direct* if $i_1 < i_2 < \cdots < i_{d'}$.



A d-permutation contains a lower dimensional pattern if one of its direct projection contains it.

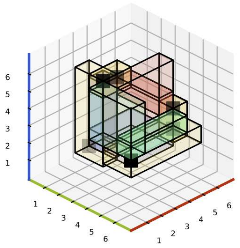
Baxter d-permutations were introduced in Bonichon & Morel 2022.

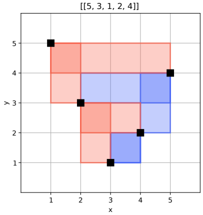
Defined as recursively well-sliced d-permutations.

Definition

A d-permutation is well sliced if each slices intersect exactly one slice of each type and two intersecting slices have the same direction.

Recursively well-sliced = projections are well-sliced.



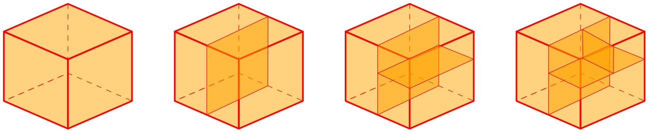


Correspond to the forbidden patterns : $\mathrm{Sym}((312, 213)|_{1,2,.})$, $\mathrm{Sym}(2413|_{2,2})$, $\mathrm{Sym}((3412, 1432)|_{2,2,.})$, $\mathrm{Sym}((2143, 1423)|_{2,2,.})$.

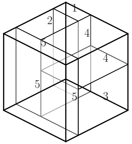
Higher dimensional floorplans

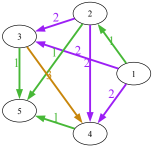
The results

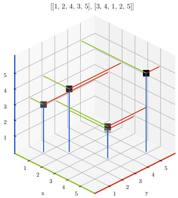
- **Definition of d-floorplans**
- **A generating tree of $d-$floorplans**



- **Bijection between $2^{d-1}-$floorplans and subset of Baxter $d-$permutations**

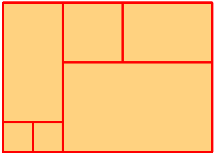


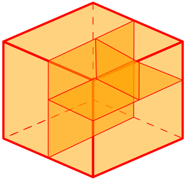




Generalizing mosaic floorplans to higher dimensions

Mosaic floorplans are made of rectangles (2D objects).





Rectangles $\rightarrow$ d-dimensional hypperrectangles (d-rectangles).

Segments (block boundaries) $\rightarrow$ $(d-1)$-dimensional hypperrectangles (block-facets) (d type of block facets, one per axis).

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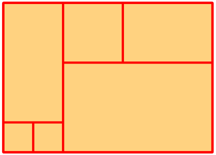

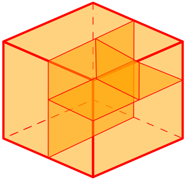

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Segments (block boundaries) $\rightarrow$ $(d-1)$-dimensional hypperrectangles (block-facets) (d type of block facets, one per axis).

Definition

A d-dimensional floorplan is a partition of a d-dimensional rectangle by smaller d-dimensional rectangles.

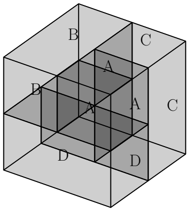
Generic condition in higher dimensions

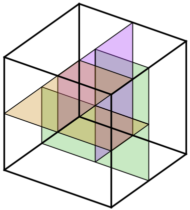
A border is the union of all block-facets at a fixed position and of the same axis type.

Definition

A d-dimensional floorplan is generic if all of its borders are $(d-1)$-rectangles.

In 2D, generic condition only forbid disconnected segments. In higher dimension there are more configurations.



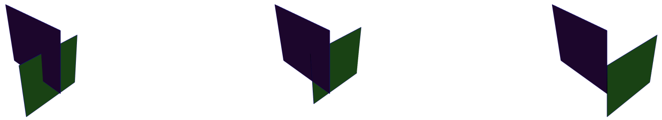


Very important to forbid them !

Tatami condition in higher dimensions

Definition

Two borders cross each other if the intersection of their interior is non empty.



The tatami condition generalizes straightforwardly in higher dimensions.

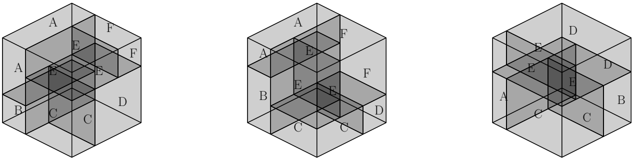
Definition

A d-dimensional floorplan fullfills the tatami condition if there are no borders crossing.

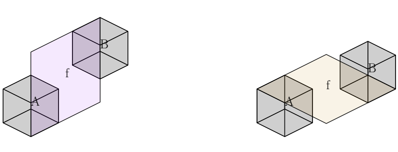
d-floorplans

Definition

A *d*-floorplan is a *d*-dimensional floorplan that fullfills both the generic and the tatami condition



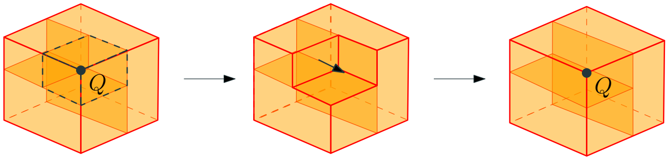
The direction relations of the blocks and the equivalence relations can also be generalized in a straightforward way. (there are now one relation per axis)



Generating tree of d-floorplans

Block deletion operation :

- Remove the block containing the top closest corner of the floorplan (corner block).
- Push the remaining border toward the boundary of the floorplan.



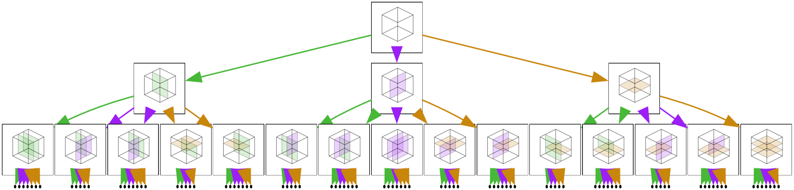
Lemma

After removing a corner block, there is a unique way to slide a border s.t. the resulting object is a floorplan with one block less.

Generating tree of d-floorplans

Block deletion define a parent-child relation between $d-$floorplans.

It gives a natural definition of a generating tree.



Encoding of the generating tree

One can encode this generating tree by :

- Associating to each floorplan a label (each label tells you how many children a floorplan has).
- Defining a rewriting rule (gives the labels of the children of a floorplan).

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For $d = 2$, the structure of the tree can be encoded by two parameters.

For $d > 2$, the structure becomes much more complex :

- The number of parameters encoding the floorplans is not bounded.
- Labels given by lists of vectors in $\mathbb{N}^d$.
- Rewriting rule given by an algorithm.

Enumeration sequences

$n \backslash d$	2	3	4	5
1	1	1	1	1
2	2	3	4	5
3	6	15	28	45
4	22	93	244	505
5	92	651	2392	6365
6	422	4917	25204	86105
7	2074	39111	278788	1221565
8	10754	322941	3193204	17932745
9	58202	2742753	37547284	270120905
10	326240	23812341	450627808	4151428385
11	1882960	210414489	5497697848	64839587065
12	11140560	1886358789	67979951368	1026189413865
13	67329992	17116221531	850063243936	
14	414499438	156900657561		
15	2593341586	1450922198319		

Not the same numbers between 2^{d-1}-floorplans and Baxter d-permutations for $d \neq 2$!

Bijection with d-permutations

Found a bijection between 2^{d-1}-floorplans and $d-$permutations avoiding $Sym(2413|_2)$ and $Sym((312, 213))$ (Sub Baxter d-permutations).

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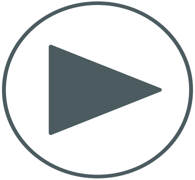
Mapping from 2^{d-1}-floorplans to d-permutations obtained through a peeling operation (recursive block deletion w.r.t a corner)



Peeling order w.r.t to d (canonical) corners gives the d-permutation.

From $d-$permutations to 2^{d-1} floorplans

To obtain a 2^{d-1}-floorplan from a sub Baxter d-permutation :

- Associate to each point p of the permutation a block b.
- Perform block insertions (inverse of block deletion) according to the relative positions of the points

Bijection with d-permutations

This bijection can also be extended to arbitrary floorplan dimensions by further avoiding d-dimensional patterns of size two.

ex : 3-floorplans and sub Baxter 3-permutations avoiding $(12, 12)$.

Objects	Permutations	Pattern avoidance
Slicing floorplans	Separable	Sym(2413)
Mosaic floorplans	Baxter	$\mathrm{Sym}(2413\|_{2,2})$
2^{d-1}-guillotine floorplans	d-Separable	$\mathrm{Sym}(2413), \mathrm{Sym}((312, 213))$
2^{d-1}-Floorplans	sub d-Baxter	$Sym(2413\|_{2})$, $\mathrm{Sym}((312, 213))$
	$d-$Baxter	$Sym(2413\|_{2})$, $\mathrm{Sym}((312, 213)\|_{1,2,.})$, $\mathrm{Sym}((3412, 1432)\|_{2,2,.})$, $\mathrm{Sym}((2143, 1423)\|_{2,2,.})$

Perspectives

Perspectives

There still remains many open questions about d-floorplans.

- More on their enumeration ?
- Other equivalence relation (strong equivalence) ?
- Study non generic d-dimensional floorplans ?
- Other generalized bijections (twin trees etc...)
- Pattern avoiding d-floorplans (studied by Asinowski and Polley for d=2)
- Look at the properties of Sub Baxter d-permutations

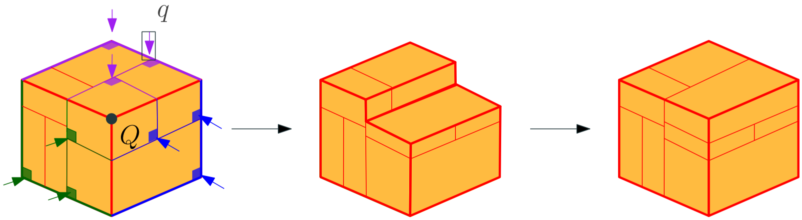
Thank you for your attention !

Block insertion

Block insertion operation : (Inverse of the block deletion)

- Choose a boundary of the $d-$floorplan containing the the top closest corner.
- Choose in it a block corner q that generates a **pushable facet**.
- Slide this facet from the boundary of the floorplan.
- Insert a block in the newly created space.

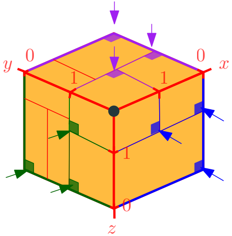
Definition : A corner q in a boundary F_j of a floorplan generates a **pushable facet** if the corners $q - Q$ define a facet in F_j (the pushable facet).



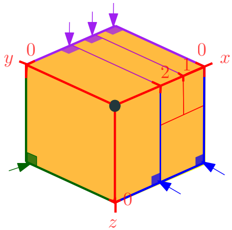
Encoding of the generating tree

Lemma : There is a unique way to obtain a given $d-$floorplan by a block insertion.

- **Number of children** of a $d-$floorplan given by its **number of pushable facets**.
- These numbers are **not enough to encode the structure of the generating tree** (rewriting rule)
- Label of a d-floorplan = **coordinates (standardized) of these corners**.



$.00, .10, .11$ $0.0, 0.1, 1.1$ $00., 01., 11.$

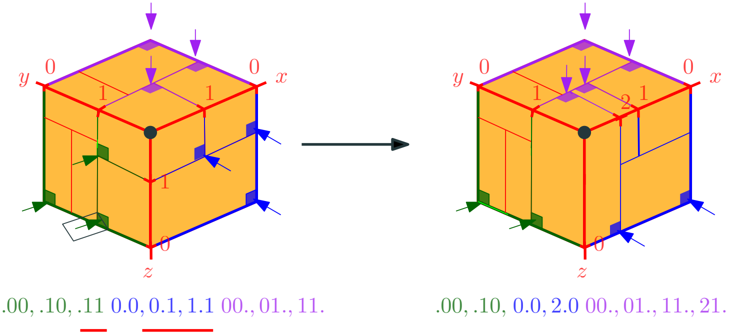


$.00$ $0.0, 2.0,$ $00., 10., 20.$

Rewriting rules of the labels

During a block insertion we :

- **Neutralize some pushable facets** → Remove some corners in the label.
- **Add a new block** → new corners in the label.



Corners removed = **corners for which one of the coordinate is greater than the one of the pushed corner**.

Bijection with $2^{D-1}-$floorplans

Definition

Let $\mathcal{P}$ be a 2^{d-1}-floorplan and let $\sigma_2, \ldots, \sigma_d$ **be the peeling orders of the blocks** with respect to the canonical corners $q_2 \ldots q_d$, ordered with respect to the order of q_1. The $d-$**permutation associated to** $\mathcal{P}$ **is defined as** $\sigma(\mathcal{P}) = (\sigma_2, \sigma_3, \ldots, \sigma_d)$**.**.

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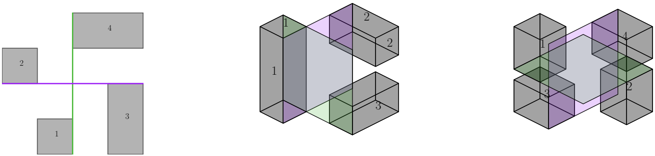
Lemma

For any 2^{d-1}-floorplan $\mathcal{P}$, the d-permutation $\sigma(\mathcal{P})$ avoids $Sym(2413|_{2,2})$, and $Sym((312, 213))$.



Lemma

The mapping from 2^{d-1}-floorplans to d-permutations is injective.

From $d-$permutations to 2^{d-1} floorplans

How to obtain a 2^{d-1}-floorplan from a $d-$permutation ?

We can define partial orders on the point of a permutation.

Definition

A direction f is an element of $\{+,-\}^d$, which is positive if first element is $+$. In a $d-$permutation, a direction between two points p_1 and p_2 is the vector $\mathbf{dir}(p_1,p_2) = (\text{sign}(x_1(p_1) - x_2(p_2)) \ldots, \text{sign}(x_d(p_1) - x_d(p_2))$.

We say that p_1 **precedes** p_2 **in the positive direction** f ($p_1 \overset{f}{\twoheadleftarrow} p_2$) if $\mathbf{dir}(p_1,p_2) = f$.