# The Insertion Encoding of Cayley Permutations

Based on joint work with Christian Bean and Paul C. Bell

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July 8, 2025

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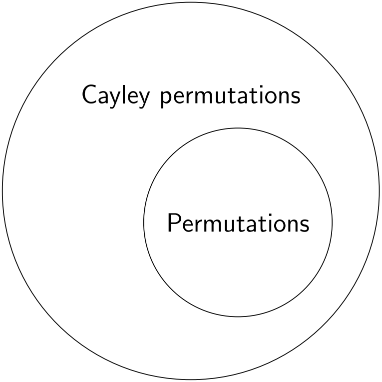
# Cayley Permutations

## Cayley Permutation

A Cayley permutation is a sequence of numbers 1 to $n$ such that every number appears at least once.

Permutations are a subset of Cayley permutations where every number appears exactly once.



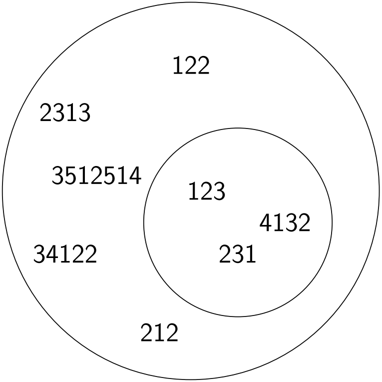
# Cayley Permutations

## Cayley Permutation

A Cayley permutation is a word $\pi \in \mathbb{N}^*$ such that every number between 1 and the maximum value of $\pi$ appears at least once.

Not Cayley permutations:

- 423
- 1253



# Ordered Set Partitions

Cayley permutations are in bijection with ordered set partitions so are counted by the Bell numbers.
The value $i$ in the $j^{th}$ block of the ordered set partition corresponds to the $i^{th}$ index of the Cayley permutation having value $j$.

$$\begin{array}{ccc} 25/14/36 & \Longrightarrow & 213213 \\ 13/26/54 & \Longrightarrow & 12133 \\ 34/25/16 & \Longrightarrow & 321123 \end{array}$$

# Cayley Permutation Classes

Pattern containment in Cayley permutations is defined in the same way as for permutations.

## Cayley permutation classes

Sets of Cayley permutations that are closed downwards with respect to pattern containment are called Cayley permutation classes.

These can be defined by the set of minimal Cayley permutations not in the class, called the basis. For a basis $B$, we denote the class of Cayley permutations avoiding $B$ as $Av(B)$.
$Av(21)$ is all non-decreasing sequences, $Av(21, 11)$ is all (strictly) increasing sequences.

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$Av(21)$ is all non-decreasing sequences, $Av(21, 11)$ is all (strictly) increasing sequences.

**Aim:** enumerate Cayley permutation classes for any given basis.

# Insertion Encoding

Insertion encoding is a language that uniquely encodes how to construct Cayley permutations by adding new maximums. It was first introduced by Albert, Linton and Ruškuc[1] for permutations.

[1] Michael Albert, Steve Linton, and Nikola Ruškuc. "The Insertion Encoding of Permutations". In: *Electr. J. Comb.* 12 (Sept. 2005). DOI: 10.37236/1944.

# Insertion Encoding

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For Cayley permutations, adapt in two different ways:

- *vertical* insertion encoding - insert new (rightmost) maxima
- *horizontal* insertion encoding - insert new rightmost values

These are the same for permutations but different for Cayley permutations.

[1] Michael Albert, Steve Linton, and Nikola Ruškuc. "The Insertion Encoding of Permutations". In: *Electr. J. Comb.* 12 (Sept. 2005). DOI: 10.37236/1944.

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- Build up Cayley permutations from configurations, e.g. $\diamond 1 \diamond 21$
- Begin with an empty slot $\diamond$.
- Slots are replaced by a combination of the next smallest, leftmost value and possibly more slots until all slots are filled.

# Vertical Insertion Encoding Example

31232

$\diamond$

# Vertical Insertion Encoding Example

31232

$\diamond 1 \diamond$

$\diamond$

# Vertical Insertion Encoding Example

31232

$\diamond 12 \diamond$

$\diamond 1 \diamond$

$\diamond$

# Vertical Insertion Encoding Example

31232

$\diamond 12 \diamond 2$

$\diamond 12 \diamond$

$\diamond 1 \diamond$

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$$\diamond$$

The whole process is called an evolution.

# Language of Encoding

Letters of the form $a_{i,j}$ where $i \in \mathbb{N}$, $j \in \{0,1\}$ and $a$ is one of $l, m, r, f$ denoting left, middle, right and fill respectively.

$$l \to n\diamond,$$

$$m \to \diamond n\diamond,$$

$$r \to \diamond n,$$

$$f \to n.$$

Where $n$ is the next number to be added.

# Vertical Insertion Encoding Example

31232

$\diamond$

# Vertical Insertion Encoding Example

31232

$\diamond 1 \diamond \quad m_{1,1}$

$\diamond$

# Vertical Insertion Encoding Example

31232

$$\begin{array}{cc} \diamond 12 \diamond & l_{2,1} \\ \diamond 1 \diamond & m_{1,1} \\ \diamond & \end{array}$$

# Vertical Insertion Encoding Example

31232

$$\begin{array}{cc} \diamond 12 \diamond 2 & r_{2,0} \\ \diamond 12 \diamond & l_{2,1} \\ \diamond 1 \diamond & m_{1,1} \\ \diamond & \end{array}$$

# Vertical Insertion Encoding Example

**31232**

$$\begin{array}{cc} 312\diamond 2 & f_{1,1} \\ \diamond 12\diamond 2 & r_{2,0} \\ \diamond 12\diamond & l_{2,1} \\ \diamond 1\diamond & m_{1,1} \\ \diamond & \end{array}$$

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$$m_{1,1} l_{2,1} r_{2,0} f_{1,1} f_{1,0} = 31232$$

# Insertion Encoding

## The Language of Insertion Encoding

For a set of Cayley permutations $C$, the words describing the insertion encoding of every element in $C$ is called the language of $C$, denoted $\mathcal{L}(C)$.

# Regular Insertion Encoding

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The insertion encoding of a Cayley permutation class is regular if it can be represented by a deterministic finite automaton for which there exists some $k \in \mathbb{N}$ such that from any state there exists a path of size $\leq k$ to an accept state.

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We can represent states as configurations and accept states as Cayley permutations. The length of a path from a configuration is at least as long as the number of slots in the configuration. For the insertion encoding to be regular, there must be a bound on the number of slots in the configurations.

## Vertical Insertion Encoding Slot Bounded

For a class, if every configuration in an evolution has no more than $k$ slots, then the class is $k$ slot bounded, denoted $\mathbf{SB}(k)$.

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$$\diamond a_1 \diamond a_2 \diamond \ldots \diamond a_k \diamond,$$

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$$b_1 a_1 b_2 a_2 b_3 \ldots b_k a_k b_{k+1}$$

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where $b_i$ to the right of $a_n$ are greater than or equal to $a_n$, and to the left are strictly greater than $a_n$.
The basis of $\mathbf{SB}(k)$ is all Cayley permutations of this form.

# Vertical Juxtapositions

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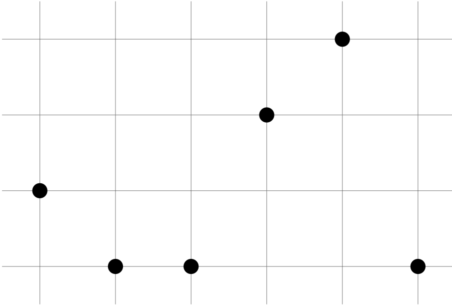
A Cayley permutation $\pi$ with maximum value $n+k$ is a *vertical juxtaposition* of a Cayley permutation $\sigma$ with maximum value $n$ and a Cayley permutation $\tau$ with maximum value $k$ if the strictly smallest $n$ values of $\pi$ standardise to $\sigma$ and the strictly largest $k$ values standardise to $\tau$.

211341 is a vertical juxtaposition of 111 and 123



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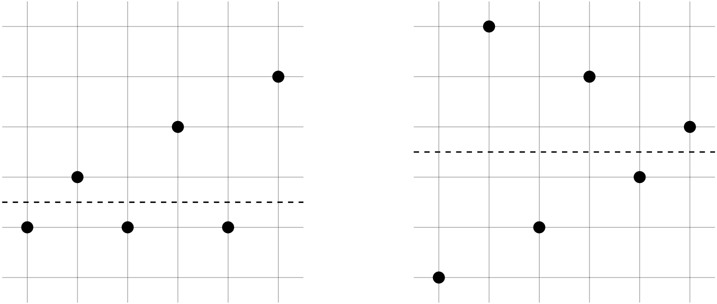
If $\sigma$ and $\tau$ are either increasing, decreasing or constant, then $\pi$ is in one of the nine classes of vertical juxtapositions $\mathcal{V}_{a,b}$ where $a$ and $b$ are one of $I$, $D$ or $C$ meaning increasing, decreasing or constant respectively.
211341 is in $\mathcal{V}_{C,I}$.

# Vertical Alternations

If the points in a vertical juxtaposition perfectly alternate between the top and bottom sequences, then it is a vertical alternation.

For example:

$121314 \in \mathcal{V}_{C,I}$ $\qquad$ $162534 \in \mathcal{V}_{I,D}$



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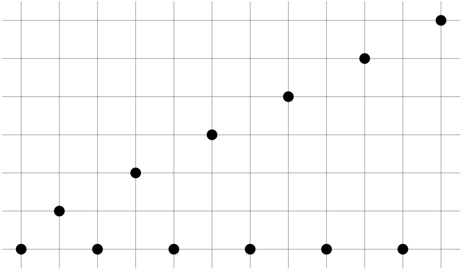
Vertical alternation 121314151617 has length 12.



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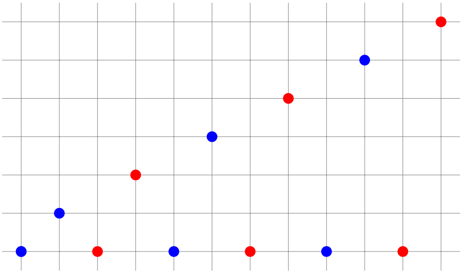

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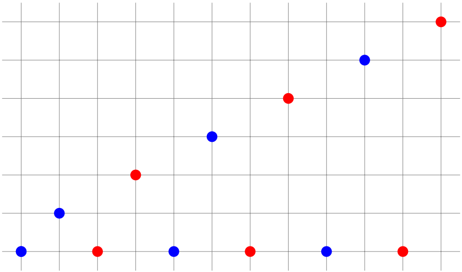

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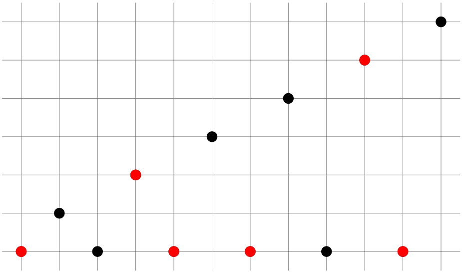

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Any vertical juxtaposition of the same type of length 6 contains one element from each pair.

# Condition to be Regular

**Theorem**

A Cayley permutation class $Av(B)$ is a subclass of $\mathbf{SB}(k)$ for some $k$ if and only if $B$ contains a Cayley permutation from each of the nine classes of vertical juxtapositions $\mathcal{V}_{I,I}$, $\mathcal{V}_{I,D}$, $\mathcal{V}_{I,C}$, $\mathcal{V}_{D,I}$, $\mathcal{V}_{D,D}$, $\mathcal{V}_{D,C}$, $\mathcal{V}_{C,I}$, $\mathcal{V}_{C,D}$ and $\mathcal{V}_{C,C}$.

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*Proof.* If $Av(B) \subseteq \mathbf{SB}(k)$, as the basis of $\mathbf{SB}(k)$ is Cayley permutations of the form

$$b_1 a_1 b_2 a_2 b_3 \ldots b_k a_k b_{k+1},$$

take $a_i$'s and $b_i$'s to be increasing, decreasing or constant sequences, then we have created all types of vertical juxtapositions.

## Condition to be Regular

Aim: There exists $k$ such that every element in the basis of $\mathbf{SB}(k)$ contains a vertical juxtaposition from B, so $Av(B) \subseteq \mathbf{SB}(k)$ for this $k$.

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- A nondecreasing sequence length $n^2$ contains a sequence length $n$ that is increasing or constant.
- Corollary: A sequence length $n^4$ contains a sequence of length $n$ that is increasing, decreasing or constant.

# Condition to be Regular

Let $n$ be the length of the longest vertical juxtaposition in $B$. Set $k = n^{16}$. For a basis element of $\mathbf{SB}(k)$ in the form

$$b_1 a_1 b_2 a_2 b_3 \ldots b_k a_k b_{k+1},$$

split into two sequences of $a$'s and $b$'s, each are length $k = n^{16}$. We can find:

- A subset of $a$'s length $n^4$ that is increasing, decreasing or constant, denote $a^*$'s.

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- A subset of $a$'s length $n^4$ that is increasing, decreasing or constant, denote $a^*$'s.
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- A subset of the $b^*$'s of length $n$ that is increasing, decreasing or constant, denote $b'$'s.

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This creates a sequence

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So, $Av(B) \subseteq \mathbf{SB}(k)$. $\square$

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So, $Av(B) \subseteq \mathbf{SB}(k)$. $\square$

## Corollary

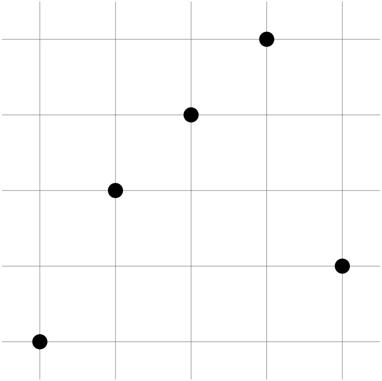
A Cayley permutation class $Av(B)$ has a regular vertical insertion encoding if and only if $B$ contains a Cayley permutation from each of the nine classes of vertical juxtapositions $\mathcal{V}_{I,I}$, $\mathcal{V}_{I,D}$, $\mathcal{V}_{I,C}$, $\mathcal{V}_{D,I}$, $\mathcal{V}_{D,D}$, $\mathcal{V}_{D,C}$, $\mathcal{V}_{C,I}$, $\mathcal{V}_{C,D}$ and $\mathcal{V}_{C,C}$.

# Horizontal juxtapositions

**Horizontal Juxtapositions**

A size $n+k$ Cayley permutation $\pi$ is a *horizontal juxtaposition* of a size $n$ Cayley permutation $\sigma$ and a size $k$ Cayley permutation $\tau$ if the size $n$ prefix of $\pi$ standardise to $\sigma$ and the size $k$ suffix of $\pi$ standardise to $\tau$.

13452 is a horizontal juxtaposition of 123 and 21



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If $\pi$ is a permutation and $\sigma$ and $\tau$ are either increasing or decreasing, then $\pi$ is in one of the four classes of horizontal juxtapositions $\mathcal{H}_{a,b}$ where $a$ and $b$ are $I$ or $D$ meaning increasing or decreasing respectively.
$13452 \in \mathcal{H}_{I,D}$.

# Condition to be regular

**Theorem**

A Cayley permutation class $Av(B)$ has a regular horizontal insertion encoding if and only if $B$ contains a Cayley permutation from each of the four classes of horizontal juxtapositions $\mathcal{H}_{I,I}$, $\mathcal{H}_{I,D}$, $\mathcal{H}_{D,I}$ and $\mathcal{H}_{D,D}$.

# Algorithm

For regular classes if a configuration is long enough we can identify and remove points which do not change whether a completed configuration contains or avoids a pattern in the basis.

Our implementation of an algorithm to do this can be found on github[2] using combinatorial exploration framework[3] to automatically discover combinatorial specifications.

---

[2] Christian Bean and Abigail Ollson. *cperms_ins_enc*. https://github.com/Ollson2921/cperms_ins_enc. 2025.

[3] Christian Bean et al. *PermutaTriangle/comb_spec_searcher: v3.0.0-zenodo*. Version v3.0.0-zenodo. June 2021. DOI: 10.5281/zenodo.4944021. URL: https://doi.org/10.5281/zenodo.4944021.

# Results

Checking if a class has a regular insertion encoding requires a linear time check. The table shows how many classes with patterns all size 3 have a regular insertion encoding of each type.

| Size of basis | Number of classes | Regular vertical | Regular horizontal | Either |
|---|---|---|---|---|
| 1 | 13 | 0 | 0 | 0 |
| 2 | 78 | 0 | 13 | 13 |
| 3 | 286 | 87 | 111 | 145 |
| 4 | 715 | 435 | 428 | 528 |
| 5 | 1287 | 1028 | 986 | 1124 |
| 6 | 1716 | 1550 | 1513 | 1625 |
| 7 | 1716 | 1645 | 1631 | 1687 |
| 8 | 1287 | 1269 | 1267 | 1283 |
| 9 | 715 | 713 | 713 | 715 |

# Stack-sorting

Cerbai[4] found the class of hare pop-stack sortable Cayley permutations to be $Av(231, 312, 2121)$. This has a regular vertical insertion encoding. Our implementation of the algorithm can be used to find the following generating function $S(x)$ for $Av(231, 312, 2121)$

$$S(x) = \frac{2x^2 - 2x^3 - x}{4x^3 - 6x^2 + 5x - 1},$$

for the sequence beginning $1, 3, 11, 41, 151, 553, 2023$ which is sequence A335793 in the OEIS[5].

[4] Giulio Cerbai. "Sorting Cayley permutations with pattern-avoiding machines". In: *Australian Journal of Combinatorics* 80 (2021), pp. 322–341.

[5] Neil J. A. Sloane. *The Online Encyclopedia of Integer Sequences*. https://oeis.org. 2025.

# Restricted growth functions

Ccayley permutations are in bijection with ordered set partitions and unordered set partitions are in bijection with a subset of Cayley permutations.

## Restricted Growth Functions

Restricted growth functions are Cayley permutations with the additional condition that for all values $k$ and $\ell$ in the Cayley permutation, if $k < \ell$ then the first occurrence of $k$ appears before the first occurrence of $\ell$.

11232 and 12323 are restricted growth functions, but 12434 is not.

Our algorithm can be adapted to compute the rational generating functions for pattern-avoiding restricted growth functions with either a horizontal or vertical regular insertion encoding.

# Thank you for listening!


- Albert, Michael, Steve Linton, and Nikola Ruškuc. "The Insertion Encoding of Permutations". In: *Electr. J. Comb.* 12 (Sept. 2005). DOI: 10.37236/1944.
- Bean, Christian and Abigail Ollson. *cperms_ins_enc*. https://github.com/Ollson2921/cperms_ins_enc. 2025.
- Bean, Christian et al. *PermutaTriangle/comb_spec_searcher: v3.0.0-zenodo*. Version v3.0.0-zenodo. June 2021. DOI: 10.5281/zenodo.4944021. URL: https://doi.org/10.5281/zenodo.4944021.
- Cerbai, Giulio. "Sorting Cayley permutations with pattern-avoiding machines". In: *Australian Journal of Combinatorics* 80 (2021), pp. 322–341.
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